Some basics
classification of reactions

**Direct reactions**
transfer momentum is small compared to initial momentum
typically peripheral
short timescale ($10^{-22}$ s)
E$>10$ MeV
mostly one step
final states keep memory of initial states

**Resonance reactions**
reactions that go through a resonance (peak in the cross section)
intermediate step in the reaction
longer time scale

**Compound reactions**
longer timescale
many steps in the reaction
all nucleons share the beam energy
loss of memory from the initial state
low energy reactions
direct reactions

Capture

$\Phi^m_{k\ell j}(\vec{r})$

Inelastic excitations (bound to bound states)

$\Phi^m_{n\ell j}(\vec{r})$ $\chi^+_{k,\sigma}(\vec{R})$ $\Phi^m'_{n'\ell' j'}(\vec{r})$

target
direct reactions

Inelastic excitations (breakup)

\[ \phi_{n\ell j}(\vec{r}) \]

\[ \chi_{\vec{k}, m} \]

\[ \chi_{\vec{k}_v, \sigma_v} \]

\[ \phi_{k\ell' j'}(\vec{r}') \]

Transfer reactions

\[ \phi_{n_i \ell_i j_i}(\vec{r}') \]

\[ \chi_{\vec{k}_i, m_i} \]

\[ \chi_{\vec{k}_f, m_f} \]

\[ \phi_{n_f \ell_f j_f}(\vec{r}') \]
why do reactions? elastic

FIG. 10. Elastic scattering for $^6$He+$^{12}$C at 38.3 MeV/nucleon in comparison with the OM results given by the real folded potential (obtained with the CDM3Y6 interaction and the Gaussian $g\sigma$ density for $^6$He). The dashed curve is obtained with the unrenormalized folded potential only. The solid curve is obtained by adding a complex surface polarization potential to the real folded potential. Its parameters, and those of the imaginary part, are explained in the text. The dotted line is obtained by folding the CDM3Y6 interaction with the compact Gaussian density $ro$.

[Lapoux et al, PRC 66 (02) 034608]
why do reactions? inelastic

traditionally used to extract electromagnetic transitions or nuclear deformations

Fig. 2. Comparison of $B(E1)$ values obtained from lifetime and Coulomb excitation measurements. The weighted average of lifetime measurements [3] (open circle) is plotted on the left along with the weighted average (solid circle) of three Coulomb excitation measurements (solid symbols). The individual Coulomb excitation measurements, GANIL (this work, square), MSU (up triangle) [6], RIKEN (down triangle) [7], and a previous GANIL experiment (diamond) [4], are plotted versus the beam energy.

[Summers et al, PLB 650 (2007) 124]
why do reactions? transfer

traditionally used to extract spin, parity and spectroscopic factors

$^{132}\text{Sn},^{133}\text{Sn})p@5\text{ MeV/u}$

why do reactions? transfer

$^{11}\text{Li}(p,t)^9\text{Li}\@ 3 \text{ A MeV}$

measured both ground state and excited state $^9\text{Li}$

[Tanihata et al, PRL 100, 192502 (2008)]
why do reactions? breakup

\[ ^{14}\text{Be} \rightarrow n+n+^{12}\text{Be} \]

two nucleon correlation function

\[ ^{23}\text{O}(^{208}\text{Pb,}^{208}\text{Pb})^{22}\text{O}+n+\gamma \]

[Nociforo et al, PLB 605 (2005) 79]

[Marques et al, PRC 64 (2001) 061301]

Fig. 1. Doppler corrected \( \gamma \)-ray spectra measured in coincidence with an \( ^{22}\text{O} \) fragment and one neutron for \( ^{208}\text{Pb} \) (symbols) and C (shaded area) targets. Arrows indicate the strongest \( \gamma \) transitions as expected from the \( ^{22}\text{O} \) level scheme of Ref. [10] (partial level scheme shown as inset; level energies are in keV).
why do reactions? fusion

Fusion of Stable vs Unstable Nuclei

Fig. 8. Reduced cross sections for the fusion of halo, normal/weakly bound, and strongly bound nuclei. (Courtesy of Kolata).

After geometric effects are scaled out, fusion enhanced for halo nuclei!

Superheavies
Halos
Applications: energy
direct reactions and tomography

The overlap function for $^{19}\text{C} \rightarrow \text{n}+^{18}\text{C}$ in arbitrary units. The radial sensitivity of the $^{18}\text{C}(d,p)^{19}\text{C}$ cross section is represented by the colored bars for different beam energies.
resonant reaction

\[ a + A \rightarrow C^* \]

Breit-Wigner shape

\[ \sigma_{\text{scatt}} = \frac{\pi}{k^2} \left(2\ell + 1\right) \frac{\Gamma^2}{(E - E_R)^2 + \Gamma^2/4} \]

\[ \Gamma = \text{FWHM} \]

\[ E_R = \text{resonant energy} \]
compound reactions

the decay of the compound state does not depend on the initial state.
classification of reactions
classification of reactions

\[ ^{56}\text{Fe} (n,n') \]
\[ E_x = 0.85 \text{ MeV} \]
selectivity of the reaction to resonances
angular distribution: compound vs direct

Direct reactions (ID):
Forward peaked (large b)

Compound reactions (NC):
Distribution is generally isotropic (except for heavy ion collision where L large)
kinematic of reactions

\[ E_{\text{tot}} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 \]

\[ E_{\text{tot}} = \frac{1}{2} m_{AB} \dot{S}^2 + \frac{1}{2} \mu \dot{R}^2 \]

energy in the relative motion

\[ E = m_B/m_{AB} \quad E_A = \frac{1}{2} \mu v_A^2. \]
kinematics of the reaction

apply laws of conservation
conservation of mass
conservation of energy
conservation of momentum

\[ m_A + m_B = m_C + m_D, \]
\[ Q + E_A + E_B = E_C + E_D, \]
\[ p_A + p_B = p_C + p_D. \]
**Figure 1.7.** Energy $E_b$ of boron nuclei in the reaction $^{12}\text{C} + ^{14}\text{N} \rightarrow ^{10}\text{I}$ as function of the energy $E_a$ of the incident $^{12}\text{C}$ nuclei, for several scattering angles.
equations of motion

\[
\left[ -\frac{\hbar^2}{2m_A} \nabla^2_{r_A} - \frac{\hbar^2}{2m_B} \nabla^2_{r_B} + V(r_A - r_B) - E_{tot} \right] \Psi(r_A, r_B) = 0.
\]

\text{Center of mass}

\[
\left[ -\frac{\hbar^2}{2m_{AB}} \nabla^2_S - \frac{\hbar^2}{2\mu} \nabla^2_R + V(R) - E_{tot} \right] \Psi(S, R) = 0.
\]

\[ \Psi(S, R) = \Phi(S) \psi(R) \]

\[ \Phi(S) = A \exp(iK \cdot S) \]

\[ -\frac{\hbar^2}{2m_{AB}} \nabla^2_S \Phi(S) = (E_{tot} - E) \Phi(S) \]

and

\[ -\frac{\hbar^2}{2\mu} \nabla^2_R + V(R) \right] \psi(R) = E \psi(R) \]
The number of particle entering a detector depends on:

• solid angular size of detector
• number of scattering centers in the target
• flux of the incident beam
• the cross sectional area for the reaction to occur

\[
\frac{dN}{dt} = j_i \ n \ \Delta \Omega \ \sigma
\]

\[
j = v |\psi|^2
\]
Cross section

Definition of cross section:
the area within which a projectile and a target will interact and give rise to a specific product.

Units 1b (barn) = 10 fm x 10 fm

If we consider just one scattering center \( n = 1 \), and measure the scattered angular flux in the final state as \( \hat{j}_f(\theta, \phi) \) particles/second/steradian, then

\[
\sigma(\theta, \phi) = \frac{\hat{j}_f(\theta, \phi)}{\hat{j}_i}
\]
Total cross section: the same in center of mass and laboratory

Angular distribution of the cross section:

\[ \sigma (\theta, \phi) \, d\phi \, \sin \theta \, d\theta = \sigma_{\text{lab}}(\theta_{\text{lab}}, \phi_{\text{lab}}) \, d\phi_{\text{lab}} \, \sin \theta_{\text{lab}} \, d\theta_{\text{lab}} \]
Scattering theory: single channel
Scattering theory: single channel

Incoming beam
\[ \psi^{\text{beam}} = A \exp(ik_i \cdot R) \]
\[ \psi^{\text{beam}} = Ae^{ik_iz} \]

Scattered wave
\[ \psi^{\text{scat}} = Af(\theta, \phi) \frac{e^{ik_f R}}{R} \]

Asymptotic wave
\[ \psi^{\text{asym}} = \psi^{\text{beam}} + \psi^{\text{scat}} = Ae^{ik_iz} + f(\theta, \phi) \frac{e^{ik_f R}}{R} \]

Incoming flux
\[ j_i = v_i |A|^2 \]

Outgoing flux
\[ j_f = v_f |A|^2 |f(\theta, \phi)|^2 / R^2 \]

Scattering amplitude
Scattering theory: single channel

Scattered angular flux and incoming flux

\[ \tilde{j}_f = R^2 j_f = v_f |A|^2 |f(\theta, \phi)|^2 \]
\[ j_i = v_i |A|^2 \]

Cross section

\[ \sigma(\theta, \phi) = \frac{v_f}{v_i} |f(\theta, \phi)|^2 \]

Renormalized scattering amplitude

\[ \tilde{f}(\theta, \phi) = \sqrt{\frac{v_f}{v_i}} f(\theta, \phi) \]
\[ \sigma(\theta, \phi) = |\tilde{f}(\theta, \phi)|^2 \]
picture for scattering

Incoming plane wave $\exp(ikz)$

Outgoing spherical waves $\exp(ikR)/R$

Beam direction $+z$
Properties of Coulomb waves with $\eta = 0$

The $\eta = 0$ functions are more directly known in terms of Bessel functions:

$$F_L(0, \rho) = \rho j_L(\rho) = (\pi \rho/2)^{1/2} J_{L+1/2}(\rho)$$

$$G_L(0, \rho) = -\rho y_L(\rho) = -(\pi \rho/2)^{1/2} Y_{L+1/2}(\rho),$$

Their behaviour near the origin, for $\rho \ll L$, is

$$F_L(0, \rho) \sim \frac{1}{(2L+1)(2L-1) \cdots 3.1} \rho^{L+1}$$

$$G_L(0, \rho) \sim (2L-1) \cdots 3.1 \rho^{-L},$$

and their asymptotic behaviour when $\rho \gg L$ is

$$F_L(0, \rho) \sim \sin(\rho - L\pi/2)$$

$$G_L(0, \rho) \sim \cos(\rho - L\pi/2)$$

$$H_L^\pm(0, \rho) \sim e^{\pm i(\rho - L\pi/2)} = i^{\mp L} e^{\pm i \rho}.$$

So $H_L^+$ describes an outgoing wave $e^{i \rho}$, and $H_L^-$ an incoming wave $e^{-i \rho}$. 
Scattering equation

- short range potentials $V(R) = 0$, $R > R_n$
  no Coulomb for now

- positive energy time-independent Schrodinger eq to obtain $f(\theta, \phi)$
  numerical solutions matched to asymptotic form

- spherical potentials $V(R) = V(R)$
  angular momentum and energy commute
  initial beam is cylindrically symm ($m=0$) implies scattered wave is too: $f(\theta, \phi) = f(\theta)$

\[
\hat{T} = -\frac{\hbar^2}{2\mu} \nabla_R^2 \\
\begin{align*}
= \frac{\hbar^2}{2\mu} & \left[ -\frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial}{\partial R} \right) + \frac{\hat{L}^2}{R^2} \right] \\
\end{align*}
\]

\[
[\hat{T} + V - E] \psi(R, \theta) = 0
\]
Solving scattering eq: overall scheme

- Solution of scattering equation needs to match onto asymptotic form

\[ [\hat{T} + V - E]\psi(R, \theta) = 0 \]

\[ \psi_{\text{asym}}(R, \theta) = e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \]
Partial wave expansion

- Legendre polynomials form a complete set
  \[ \sum_L b_L(R) P_L(\cos \theta) \]

- they are eigenstates of \( \hat{L}^2 \) and \( \hat{L}_z \)

- orthogonality relation:
  \[ \int_0^{\pi} P_L(\cos \theta) P_{L'}(\cos \theta) \sin \theta \, d\theta = \frac{2}{2L+1} \delta_{LL'} \]

- particular form for expansion
  \[ \nabla^2_R P_L(\cos \theta) \frac{\chi_L(R)}{R} = \frac{1}{R} \left( \frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right) \chi_L(R) P_L(\cos \theta) \]

- partial wave expansion:
  \[ \psi(R, \theta) = \sum_{L=0}^{\infty} (2L+1) i^L P_L(\cos \theta) \frac{1}{kR} \chi_L(R) \]

- partial wave equation:
  \[ \left[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right) + V(R) - E \right] \chi_L(R) = 0 \]
Matching to asymptotics

- 2\textsuperscript{nd} order differential equation $\rightarrow$ two boundary conditions

1. for \textit{wfn} to be finite everywhere
   \[ \chi_L(0) = 0 \]
2. asymptotically
   \[ \psi_{\text{asym}}(R, \theta) = e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \]

\[ \left[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right) + V(R) - E \right] \chi_L(R) = 0 \]

\[ \psi_{\text{asym}}(R, \theta) = e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \]
free solution and coulomb functions

- when \( V(R) = 0 \), for all \( R \)

\[
\left[ \frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} + 1 \right] \chi_L^{\text{ext}}(\rho/k) = 0
\]

- Coulomb wave equation

\[
\left[ \frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1 \right] X_L(\eta, \rho) = 0
\]

- two linearly independent solutions:
  - regular and irregular Coulomb functions
  \( F_L(\eta, \rho) \quad G_L(\eta, \rho) \)

- two linearly independent solutions:
  - outgoing and incoming Hankel functions
  \( H_L^{\pm} = G_L \pm iF_L \)
Partial wave expansion for plane wave

\[
e^{ikz} = \sum_{L=0}^{\infty} (2L+1)i^LP_L(\cos \theta) \frac{1}{kR} F_L(0, kR)
\]

\[
e^{ikz} = \sum_{L=0}^{\infty} (2L+1)i^LP_L(\cos \theta) \frac{1}{kR} \frac{i}{2} \left[ H_L^- (0, kR) - H_L^+ (0, kR) \right]
\]

- at large distances the radial wave function should behave as

\[
\chi_L^{\text{ext}} (R) = A_L \left[ H_L^- (0, kR) - \mathbf{S}_L H_L^+ (0, kR) \right]
\]

incoming \hspace{2cm} \text{outgoing}

partial wave S-matrix element
Matching to asymptotics

- numerical solution is proportional to true solution

\[ \chi_L(R) = B u_L(R) \]

\[ u''_L(R) = \left[ \frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2}(V(R) - E) \right] u_L(R) \]

\[ B u_L(R) = \chi_L(R) \xrightarrow{R \to R_n} \chi_L^{\text{ext}}(R) = A_L \left[ H_L^-(0, kR) - S_L H_L^+(0, kR) \right] \]

\[ u_L(0) = 0 \]

\[ u'_L(0) \neq 0 \]

\[ a > R_n \]
The matching can be done with the inverse log derivative $R_L$

- any potential will produce $R_L$ which relates to $S_L$

\[
R_L = \frac{1}{a} \frac{\chi_L'(a)}{\chi_L(a)} = \frac{1}{a} \frac{u_L'(a)}{u_L(a)}
\]

\[
R_L = \frac{1}{a} \frac{H_L^- - S_L H_L^+}{H_L^+ - aR_L H_L'}
\]

- $u_L(0) = 0$
- $u_L'(0) \neq 0$

$a > R_n$
S-matrix and scattering amplitude

- to obtain the scattering amplitude need to sum the partial waves

\[ \psi(R, \theta) \xrightarrow{R \gg R_n} \frac{1}{kR} \sum_{L=0}^{\infty} (2L+1) i^L P_L(\cos \theta) A_L [H_L^-(0, kR) - S_L H_L^+(0, kR)] \]

\[ \psi^{\text{asym}}(R, \theta) = e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \]

- homework 6!

\[ f(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) (S_L - 1) \]

\[ \sigma(\theta) \equiv \frac{d\sigma}{d\Omega} = \left| \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) (S_L - 1) \right|^2 \]
Phase shifts

- Each partial wave S-matrix can be equivalently described with a phase shift $S_L = e^{2i\delta_L}$

$$\delta_L(E) = \frac{1}{2i} \ln S_L + n(E)\pi$$

added to make the phase shift continuous

- Scattering amplitude in terms of phase shifts

$$f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) e^{i\delta_L} \sin \delta_L$$

- Asymptotic form in terms of phase shift

$$\chi^\text{ext}_L(R) \rightarrow e^{i\delta_L} [\cos \delta_L \sin(kR-L\pi/2) + \sin \delta_L \cos(kR-L\pi/2)]$$

$$= e^{i\delta_L} \sin(kR + \delta_L - L\pi/2).$$
Phase shifts as a function of energy

- attractive potentials: \( \delta > 0 \)
- repulsive potentials: \( \delta < 0 \)
The partial wave T-matrix is defined as the amplitude of the outgoing wave:

\[ \chi_L^{\text{ext}}(R) = F_L(0, kR) + T_L H_L^+(0, kR) \]

Simple relation with the scattering amplitude:

\[ S_L = 1 + 2i T_L \]

Simple relation with the scattering amplitude:

\[ f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) T_L \]
Relations between $T$, $S$, $\delta$

Using:

$$\chi(R) = e^{i\delta}[F \cos \delta + G \sin \delta]$$

$$\delta = \delta$$

$$T = e^{i\delta} \sin \delta$$

$$S = e^{2i\delta}$$

$$V = 0 \quad \delta = 0$$

$$V \text{ real} \quad \delta \text{ real}$$

$$T = 0 \quad S = 1$$

$$|1 + 2iT| = 1 \quad |S| = 1$$

$$F + T H^+ \quad \frac{i}{2}[H^- - S H^+]$$

$$\arctan \frac{T}{1 + iT} \quad \frac{1}{2i} \ln S$$

$$\frac{1}{2} (1 - S)$$
Integrated cross sections

- use properties of Legendre polynomials

\[
\sigma_{el} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \, \sigma (\theta)
\]

\[
= 2\pi \int_0^\pi d\theta \sin \theta |f (\theta)|^2
\]

\[
= \frac{\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) |1 - S_L|^2
\]

\[
= \frac{4\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L.
\]

- Optical theorem: total elastic cross section related to zero-angle scattering amplitude

\[
f (0) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1)(e^{2i\delta_L} - 1),
\]

\[
\text{Im} f (0) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L
\]

\[
= \frac{k}{4\pi} \sigma_{el}.
\]
Resonances and phase shifts

- particles trapped inside a barrier

\[ \Gamma \sim \frac{\hbar}{\tau} \]

- Resonance characterized by J, E, \( \Gamma > 0 \)

- will show rapid rise of phase shift

\[ \Delta t \sim \frac{\hbar d}{dE} \delta(E) \]

- there is usually a background in addition to the resonance part:

\[ \delta(E) = \delta_{bg}(E) + \delta_{res}(E) \]

\[ \delta_{res}(E) = \arctan \left( \frac{\Gamma/2}{E_r - E} \right) + n(E)\pi \]

- in a pure case, with no background at the resonance energy \( \delta = \pi/2 \)
Resonances and cross sections

- Breit-Wigner form

\[
\sigma_{el}^{\text{res}}(E) \simeq \frac{4\pi}{k^2} (2L+1) \sin^2 \delta_{\text{res}}(E) \\
= \frac{4\pi}{k^2} (2L+1) \frac{\Gamma^2/4}{(E - E_r)^2 + \Gamma^2/4}
\]
Fig. 3.2. Examples of resonant phase shifts for the $J^\pi = 3/2^-$ channel in low-energy n-$\alpha$ scattering, with a pole at $E = 0.96 - i0.92/2$ MeV. There is only a hint of a resonance in the phase shifts for the $J^\pi = 1/2^-$ channel, but it does have a wide resonant pole at $1.9 - i6.1/2$ MeV.
Fig. 3.3. Possible Breit-Wigner resonances. The upper panel shows resonant phase shifts with several background phase shifts $\delta_{bg} = 0, \pi/4, \pi/2$ and $3\pi/4$ in the same partial wave. The lower panel gives the corresponding contributions to the total elastic scattering cross section from that partial wave.
Resonances and S-matrix

- S-matrix form around the resonance

\[
S(E) = e^{2i\delta_{bg}(E)} \frac{E - E_r - i\Gamma/2}{E - E_r + i\Gamma/2}
\]

- if analytic continuation to complex energies
  S-matrix pole at \( E_p = E_r - i\Gamma/2 \)
Fig. 3.4. The correspondences between the energy (left) and momentum (right) complex planes. The arrows show the trajectory of a bound state caused by a progressively weaker potential: it becomes a resonance for $L > 0$ or when there is a Coulomb barrier, otherwise it becomes a virtual state. Because $E \propto k^2$, bound states on the positive imaginary $k$ axis and virtual states on the negative imaginary axis both map onto the negative energy axis.
Virtual states

- neutral L=0 particles: no barrier
- S-matrix pole is on negative imaginary k-axis (not a bound state!)
- scattering length
  \[ k_p = \frac{i}{a_0} \]
- S-matrix in terms of scattering length
  \[ S(k) = -\frac{k + \frac{i}{a_0}}{k - \frac{i}{a_0}} \]
- phase shift in terms of scattering length
  \[ k \cot \delta(k) = -\frac{1}{a_0} \]