Scattering theory II: continuation

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What we learnt?

Scattering amplitude nuclear only

\[ f(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) (s_L - 1) \]

\[ \sigma(\theta) \equiv \frac{d\sigma}{d\Omega} = \left| \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) (s_L - 1) \right|^2 \]

Coulomb+nuclear

\[ f_n(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) e^{2i\sigma_L(\eta)} (s_L^n - 1) \]

\[ \sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 \equiv |f_{nc}(\theta)|^2 \]

Integrated cross sections:

\[ \sigma_{el} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \sigma(\theta) \]

\[ = 2\pi \int_0^{\pi} d\theta \sin \theta |f(\theta)|^2 \]

\[ = \frac{\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) |1 - s_L|^2 \]

\[ = \frac{4\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L, \]

\[ \sigma_A = \frac{2}{h\nu} \frac{4\pi}{k^2} \sum_{L} (2L+1) \int_0^{\infty} \left[ -W(R) \right] |\chi_L(R)|^2 dR \]

\[ \sigma_R = \frac{\pi}{k^2} \sum_{L} (2L+1)(1 - |s_L|^2) \]
Optical potentials

- obtained from:
  1) Fitting a single elastic scattering data set (local optical potential)
  2) Fitting many sets of elastic data at several energies on several targets (global optical potential)
  3) Theory (folding models – depend on density distribution)

- real parts get weak with beam energy (become repulsive at 300 MeV)
- imaginary terms dominate at the higher energies

- Coulomb interaction: uniform charge distribution with radius $R_{\text{coul}}$

\[
V_{\text{Coul}}(R) = Z_p Z e^2 \times \left\{ \begin{array}{ll}
\left( \frac{3}{2} - \frac{R^2}{2 R_{\text{coul}}^2} \right) \frac{1}{R_{\text{coul}}} & \text{for } R \leq R_{\text{coul}} \\
\frac{1}{R} & \text{for } R \geq R_{\text{coul}}
\end{array} \right.
\]
Optical potentials

- real part weaker for neutrons than for protons

\[ V(R) = V_0(R) + \frac{1}{2} t_z \frac{N - Z}{A} V_T(R) \]

isoscalar and isovector components

- spin orbit term: couples spin and orbital motion
elastic scattering in fresco

```
p+Ni78 Coulomb and Nuclear;
NAMELIST
 &FRESCO hcm=0.1 rmatch=60
     jtmin=0.0 jtmax=50 absend= 0.0010
     thmin=0.00 thmax=180.00 thinc=1.00
     chans=1 smats=2 xstabl=1
     elab(1:3)=6.9 11.00 49.350 nlab(1:3)=1 1 /

 &PARTITION namep='p' massp=1.00 zp=1
     namet='Ni78' masst=78.0000 zt=28 qval=-0.000 nex=1 /
 &STATES jp=0.5 bandp=1 ep=0.0000 cpot=1 jt=0.0 bandt=1 et=0.0000 /
 &partition /

 &POT kp=1 ap=1.000 at=78.000 rc=1.2 /
 &POT kp=1 type=1 p1=40.00 p2=1.2 p3=0.65 p4=10.0 p5=1.2 p6=0.500 /
 &pot /
 &overlap /
 &coupling /
```

Box B.1 FRESCO input for the elastic scattering of protons on $^{78}$Ni at several beam energies
Which curve corresponds to highest energy?
Direct and exchange amplitudes

- so far we have considered $p+ t \rightarrow p' + t'$
- now we need to consider:

(a) Scattering of identical fermions: $p = t$ of odd baryon number;
(b) Scattering of identical bosons: $p = t$ of even baryon number; and
(c) Exchange scattering: $p' = t$ and $t' = p$, and $p$ is distinguishable from $t$, 

- consider an exchange index $\varepsilon = +1$ for boson-boson
  $\varepsilon = -1$ for fermion-fermion

$$\hat{P}_{pt} \Psi^{M_{\text{tot}}}_{xJ_{\text{tot}}} (R_x, \xi_p, \xi_t) = \varepsilon \Psi^{M_{\text{tot}}}_{xJ_{\text{tot}}} (-R_x, \xi_t, \xi_p)$$

$$\varepsilon = (-1)^{2I_p}$$
Direct and exchange amplitudes

- For two identical spinless particles:

\[ \psi_{\text{asy}m}(R) = A \left[ e^{ikz} + f(\theta) \frac{e^{ikR}}{R} \right] \]

- For two identical particles the wfns should be:

\[ \Psi_{\text{asy}m}^\varepsilon(R) = \psi_{\text{asy}m}(R) + \varepsilon \psi_{\text{asy}m}(-R) \]

- Scattered outgoing wave properly symmetrized should be:

\[ \Psi_{\text{out}}^\varepsilon(R) = A[f(\theta) + \varepsilon f(\pi - \theta)] \frac{e^{ikR}}{R} = Af_{\varepsilon}(\theta) \frac{e^{ikR}}{R} \]
Direct and exchange amplitudes

- Cross section for identical particle scattering

\[
\sigma(\theta) = |f_\varepsilon(\theta)|^2 = |f(\theta) + \varepsilon f(\pi - \theta)|^2 = |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2\varepsilon \Re f(\theta)^*f(\pi - \theta).
\]

- The partial wave expansion for the scattering amplitude is:

\[
f_\varepsilon(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1)P_L(\cos \theta) T_L[1 + \varepsilon (-1)^L].
\]

For bosons, odd partial waves do not contribute!
Even partial waves are doubled!
Direct and exchange with spin

- permutation best done in LS coupling:

\[
\hat{P}_{pt} |L(I_p, I_t)S; J_{\text{tot}x}\rangle = (-1)^L (-1)^{S-I_p-I_t} |L(I_t, I_p)S; J_{\text{tot}x}\rangle
\]

- the partial wave expansion for the scattering amplitude is:

\[
f_S(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) T_L [1 + \varepsilon (-1)^{L+S-I_p-I_t}]
\]

\[
= \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) T_L [1 + (-1)^{L+S}]
\]

For S=0 odd partial waves do not contribute!
For S=1 even partial waves do not contribute!
Direct and exchange with spin

- characteristic interference patterns for different spin states!

![Graphs showing cross section vs. theta](image)

Fig. 3.9. Fermion singlet (a) and triplet (b) nucleon-nucleon scattering cross sections, assuming pure Coulomb scattering with $\eta = 5$. Case (a) also applies for boson scattering. The cross section is in units of $\eta^2/4k^2$. 
Fitting data

Comparing theory and experiment
\( \{p_j\} \) inputs parameter set
\( \sigma^{\text{exp}}(i) \) experimental data (\( \Delta \sigma \) standard deviation)

Measure of discrepancy

\[
\chi^2 = \sum_{i=1}^{N} \frac{(\sigma^{\text{th}}(i) - \sigma^{\text{exp}}(i))^2}{\Delta \sigma(i)^2}.
\]

If theory agrees exactly with experiment \( \chi^2 = 0 \) (very unlikely!)

What is statistically reasonable \( \sigma^{\text{th}}(i) - \sigma^{\text{exp}}(i) \sim \Delta \sigma(i) \) so \( \chi^2 \sim N \) (or \( \chi^2/N \sim 1 \))

If \( \chi^2/N >> 1 \) then theory needs improvement
If \( \chi^2/N << 1 \) errors have been overestimated
Fitting data (including an overall scale)

Comparing theory and experiment
\{p_j, s\} inputs parameter set (s is overall scale)
\sigma^{\text{exp}}(i) experimental data (\Delta\sigma \text{ standard deviation})

Measure of discrepancy

\[ \chi^2 = \frac{(s - E[s])^2}{\Delta s^2} + \sum_{i=1}^{N} \frac{(\sigma^{\text{th}}(i) - s \sigma^{\text{exp}}(i))^2}{\Delta \sigma(i)^2} \]

If theory agrees exactly with experiment \( \chi^2 = 0 \) (very unlikely!)
What is statistically reasonable \( \sigma^{\text{th}}(i) - \sigma^{\text{exp}}(i) \sim \Delta \sigma(i) \) so \( \chi^2 \sim N + 1 \) (or \( \chi^2/(N+1) \sim 1 \))
Expanding chi2 around a minimum

Let us consider the expansion of chi2 around a minimum found for the set of parameters \( \{ p_j^0 \} \)

\[
\chi^2(p_1, \ldots, p_P) \approx \chi^2(p_1^0, \ldots, p_P^0) + \frac{1}{2} \sum_{m,n=1}^{P} H_{mn}(p_m - p_m^0)(p_n - p_n^0) \\
\equiv \chi^2(p^0) + \frac{1}{2}(p - p^0)^T H (p - p^0)
\]

Hesse matrix

\[
H_{mn} = \frac{\partial^2}{\partial p_m \partial p_n} \chi^2(p_1, \ldots, p_P)
\]

Covariance matrix

\[
(V^p)^{-1} = \frac{1}{2}H \quad \text{or} \quad V^p = 2H^{-1}.
\]

The fitting probability can be defined in terms of the Hesse matrix:

\[
P_{tot} = \frac{1}{(2\pi)^{N/2} \Delta} e^{-\frac{\chi^2(p^0)}{2}} \exp \left[ -\frac{1}{4} \sum_{mn}^P (p_m - p_m^0) H_{mn}(p_n - p_n^0) \right]
\]
Allowed parameters within 1sigma

This happens with argument of exp is 1/2

\[ \frac{1}{2} \sum_{mn}^{P} (p_m - p_m^0) H_{mn} (p_n - p_n^0) = 1 \]

Using the Taylor expansion this can be written as

\[ \chi^2(p_1, \ldots, p_P) = \chi^2(p_1^0, \ldots, p_P^0) + 1 \]
Ex: optical potential fits (optical model)

- Strongly non linear (fitting is done by iteration only)
- Need data at large scattering angles
- Spin orbit does not strongly affect elastic cross sections
  - Ambiguities:
    - Low energy (phase equivalent potentials)
    - Medium energy (volume integral $V_{ws} = R_{ws}^2$)
    - Heavy nuclei (governed by tail of $V$)

\[
\mathcal{J} = \int V(r) dr = 4\pi \int_0^\infty V(r) r^2 dr
\]

\[
V(R) \approx -V_{ws} e^{-(R-R_{ws})/a_{ws}} = -V_{ws} e^{R_{ws}/a_{ws}} e^{-R/a_{ws}}
\]
Strategies for chi2 fitting

• Start with simplest data and simplest reaction model
  (for example elastic and optical model)

• Restart from any intermediate stage

• If there are ambiguities, do grid searches and look at correlations in errors

• Artificially reduce error in data points if theory is having a hard time to get close in some region

• If minimum is found near the end of the range of a parameter, this is spurious – repeat with wider range

• Constrain with other experiments

• Two correlated variables: combine into one

Progressive improvement policy
elastic scattering fitting: examples

$^{132}\text{Sn}(d,d)@5$ MeV/u

\begin{align*}
\frac{d\sigma}{d\Omega} (\text{ratio to Rutherford}) \\
\theta (\text{degrees})
\end{align*}

$^{132}\text{Sn}(d,p)^{133}\text{Sn}@5$ MeV/u

\begin{align*}
\frac{d\sigma}{d\Omega} (\text{mb/srad}) \\
\theta (\text{degrees})
\end{align*}
elastic scattering fitting: examples

$^{48}\text{Ca}(d,d)@20$ MeV

Lovell et al.
Scattering theory III: integral forms
integral equations: green’s function methods

\[ \psi_{\alpha}(R) = \delta_{\alpha\alpha_i} F_{\alpha}(R) + \frac{2\mu_x}{\hbar^2} \int G^+(R, R') \Omega_{\alpha}(R') dR', \]

Last class we worked out the explicit form for \( G \) in coordinate space

Also we used the general asymptotic in terms of the T-matrix

\[ \psi_{\alpha\alpha_i}(R) = F_{\alpha}(R) \delta_{\alpha\alpha_i} + H_{\alpha}^+(R) T_{\alpha\alpha_i} \]

To obtain:

\[ T_{\alpha\alpha_i} = -\frac{2\mu_x}{\hbar^2 k_{\alpha}} \langle F^*_{\alpha} | \Omega_{\alpha} \rangle = -\frac{2\mu_x}{\hbar^2 k_{\alpha}} \langle F^{(-)}_{\alpha} | \Omega_{\alpha} \rangle. \]
Formal solutions to Scattering

Split the Hamiltonian in Free Hamiltonian and residual interaction

- any other interaction can in principle be include in $H_0$
- $V$ should be short range

Free scattering equation: homogeneous

\[
(H_0 - E)\phi_k(r) = 0
\]

\[
\int \phi_{k'}^*(r)\phi_k(r) \, dr = \delta(k - k')
\]

\[
\int \phi_{k}^*(r')\phi_k(r) \, dk = \delta(r - r')
\]

General Scattering equation: inhomogeneous

\[
(H_0 - E)\psi_k^\pm(r) = -V\psi_k^\pm(r)
\]

Solution can be expressed as:

\[
\psi_k^+(r) = \phi_k(r) + \frac{2\mu}{\hbar^2} \int G^+(r, r')V(r')\psi_k^+(r') \, dr'
\]

Where the Green’s function is solution of:

\[
(H_0 - E)G^+(r, r') = -\frac{\hbar^2}{2\mu}\delta(r - r')
\]
Lippmann-Schwinger Equation

Rewriting in short form: \[ \psi_k^+ = \phi_k + G^+ V \psi_k^+ \]

where the Green’s function operator is related to the Green’s function by

\[ G^+(r, r') = \langle r | G^+ | r' \rangle \]

The Green’s function operator can be expressed by

\[ G^+ = \lim_{\epsilon \to 0} \frac{1}{E - H_0 + i\epsilon} \]

Lippmann-Schwinger integral equation:

\[ \psi_k^+ = \phi_k + \frac{1}{E - H_0 + i\epsilon} V \psi_k^+ \]

equivalent to the differential form:

\[ (E - H_0)\psi_k^+ = (E - H_0)\phi_k + V \psi_k^+ \]

Born series expansion

\[ \psi_k^+ = \phi_k + G^+ V (\phi_k + G^+ V \psi_k^+) \]

\[ = \phi_k + G^+ V \phi_k + G^+ V G^+ V (\phi_k + G^+ V \psi_k^+) \]

\[ = \left( 1 + \sum_{n=1}^{\infty} (G^+ V)^n \right) \phi_k \]
Define a transition matrix (t-matrix) such that:

\[
\langle \phi_{k'} | t | \phi_k \rangle = \langle \phi_{k'} | V | \psi_k^+ \rangle
\]

Remember the Born series?

\[
\psi_k^+ = \phi_k + G^+ V (\phi_k + G^+ V \psi_k^+ )
\]

\[
= \phi_k + G^+ V \phi_k + G^+ V G^+ V (\phi_k + G^+ V \psi_k^+ )
\]

\[
= (1 + \sum_{n=1}^{\infty} (G^+ V)^n ) \phi_k
\]

Multiply by: \( \langle \phi_{k'} | V \rangle \)

and we can obtain an operator form of the equation in terms of the t-matrix

\[
t = V (1 + \sum_{n=1}^{\infty} (G^+ V)^n )
\]

\[
t = V + V G^+ t
\]

often used in few-body methods
Integral forms and T-matrix approach

\[ \psi = \phi + \hat{G}^+ \Omega \]
\[ = \phi + \hat{G}^+ V \psi, \]

**Lippmann-Schwinger equation**

\( \phi \) is incoming free wave (only non zero for elastic channel)

\( \psi \) is full wavefunction

\[ T = -\frac{2\mu}{\hbar^2 k} \langle \phi^- | V | \psi \rangle \equiv -\frac{2\mu}{\hbar^2 k} \int \phi(R) V(R) \psi(R) dR. \]

\[ T(k', k) = \langle e^{i k' \cdot R} | V | \Psi(R; k) \rangle. \]

\[ f(k'; k) = -\frac{\mu}{2\pi \hbar^2} T(k', k) \]
two potential formula: definitions

Consider your potential can be split into two parts: $U = U_1 + U_2$

**Free:**

\[ [E - T] \phi = 0 \]

\[ \hat{G}_0^+ = [E - T]^{-1} \]

\[ \phi = F \]

**Distorted:**

\[ [E - T - U_1] \chi = 0 \]

\[ \chi = \phi + \hat{G}_0^+ U_1 \chi \]

\[ \chi \rightarrow \phi + T^{(1)} H^+ \]

**Full:**

\[ [E - T - U_1 - U_2] \psi = 0 \]

\[ \psi = \phi + \hat{G}_0^+ (U_1 + U_2) \psi \]

\[ \psi \rightarrow \phi + T^{(1+2)} H^+ \]
two potential formula: derivation 1

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<td>( \psi = \phi + \hat{G}_0^+(U_1 + U_2)\psi )</td>
<td>( \psi \rightarrow \phi + T^{(1+2)} H^+ )</td>
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\[
- \frac{\hbar^2 k}{2\mu} T^{(1+2)} = \int \phi (U_1 + U_2) \psi \, dR
\]

\[
= \int (\chi - \hat{G}_0^+ U_1 \chi) (U_1 + U_2) \psi \, dR
\]

\[
= \int \left[ \chi (U_1 + U_2) \psi - (\hat{G}_0^+ U_1 \chi) (U_1 + U_2) \psi \right] dR.
\]
two potential formula: derivation 2

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<th>Free:</th>
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<td>$[E-T-U_1-U_2]\psi = 0$</td>
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<td>$\psi \to \phi + T^{(1+2)} H^+$</td>
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\[ -\frac{\hbar^2 k}{2\mu} T^{(1+2)} = \int [\chi (U_1 + U_2)\psi - \chi U_1 \hat{G}_0^+ (U_1 + U_2)\psi] \, dR \]

\[ = \int [\chi (U_1 + U_2)\psi - \chi U_1 (\psi - \phi)] \, dR \]

\[ = \int [\phi U_1 \chi + \chi U_2 \psi] \, dR \]

\[ = \langle \phi^{(-)} | U_1 | \chi \rangle + \langle \chi^{(-)} | U_2 | \psi \rangle. \]
two potential formula: result

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\[
T^{(1+2)} = T^{(1)} + T^{2(1)}
\]

\[
T^{2(1)} = -\frac{2\mu}{\hbar^2 k} \int \chi U_2 \psi \ dR
\]
\[
\chi = \phi + \hat{G}_0^+ U [\phi + \hat{G}_0^+ U [\phi + \hat{G}_0^+ U [\cdots]]] \\
= \phi + \hat{G}_0^+ U \phi + \hat{G}_0^+ U \hat{G}_0^+ U \phi + \hat{G}_0^+ U \hat{G}_0^+ U \phi \hat{G}_0^+ U \phi + \cdots ,
\]

\[
T = -\frac{2\mu}{\hbar^2 k} \left[ \langle \phi^{(-)} | U | \phi \rangle + \langle \phi^{(-)} | U \hat{G}_0^+ U | \phi \rangle + \cdots \right].
\]
plane wave Born approximation (PWBA)

$$T = -\frac{2\mu}{\hbar^2 k} \left[ \langle \phi^- | U | \phi \rangle + \langle \phi^- | U \hat{G}_0^+ U | \phi \rangle + \cdots \right].$$

$$T_{PWBA} = -\frac{2\mu}{\hbar^2 k} \langle \phi^- | U | \phi \rangle.$$

$$T_{PWBA}^L = -\frac{2\mu}{\hbar^2 k} \int_0^\infty F_L(0, kR) U(R) F_L(0, k_R) \, dR.$$

$$f_{PWBA}(\theta) = -\frac{\mu}{2\pi \hbar^2} \int dR \, e^{-i\mathbf{q}\cdot\mathbf{R}} U(\mathbf{R})$$
two potential scattering: post

\[ T^{(1+2)} = T^{(1)} - \frac{2\mu}{\hbar^2 k} \langle \chi^(-) | U_2 | \psi \rangle \]

\[ T^{(1+2)} = T^{(1)} - \frac{2\mu}{\hbar^2 k} \left[ \langle \chi^(-) | U_2 | \chi \rangle + \langle \chi^(-) | U_2 \hat{G}_1 U_2 | \chi \rangle + \cdots \right]. \]

If \( U_2 \) is weak we might expect the series to converge
two potential scattering: post and prior

\[ T^{(1+2)} = T^{(1)} - \frac{2\mu}{\hbar^2 k} \langle \chi^{(-)}|U_2|\psi \rangle \]

If \( U_2 \) is weak we might expect the series to converge

\[ T^{(1+2)} = T^{(1)} - \frac{2\mu}{\hbar^2 k} \langle \psi^{(-)}|U_2|\chi \rangle. \]

\[ T^{(1+2)}_{\alpha\alpha_i} = T^{(1)}_{\alpha\alpha_i} - \frac{2\mu_\alpha}{\hbar^2 k_\alpha} \langle \chi^{(-)}_{\alpha}|U_2|\psi^{(+)}_{\alpha_i} \rangle \quad \text{[post]}, \]

\[ = T^{(1)}_{\alpha\alpha_i} - \frac{2\mu_\alpha}{\hbar^2 k_\alpha} \langle \psi^{(-)}_{\alpha}|U_2|\chi^{(+)}_{\alpha_i} \rangle \quad \text{[prior]}. \]
distorted wave Born approximation (DWBA)

Born series is truncated after the first term

\[ T_{\text{DWBA}}^{\text{1st}} = T^{(1)} - \frac{2\mu}{\hbar^2 k} \langle \chi^{(-)} | U_2 | \chi \rangle \]

$U_2$ appears to first order

There is similarly a second-order DWBA expression

\[ T_{\alpha\alpha_i}^{\text{2nd-DWBA}} = \frac{2\mu_\alpha}{\hbar^2 k_\alpha} \left[ \langle \chi_\alpha^{(-)} | U_2 | \chi_{\alpha_i} \rangle + \langle \chi_\alpha^{(-)} | U_2 \hat{G}_1^+ U_2 | \chi_{\alpha_i} \rangle \right] . \]

$U_2$ appears to second order
multiple orders in DWBA

\[ T_{\alpha \alpha_i}^{2nd-DWBA} = -\frac{2\mu_\alpha}{\hbar^2 k_\alpha} \left( \langle \chi_\alpha^{(-)} | U_2 | \chi_{\alpha_i} \rangle + \langle \chi_\alpha^{(-)} | U_2 \hat{G}_1^+ U_2 | \chi_{\alpha_i} \rangle \right) . \]
Methods for solving the problem

Differential equations:
- Direct integration methods (Numerov, Runge-Kutta)
- Iterative methods
- R-matrix methods
- Other expansion methods transforming the problem into a diagonalization problem (Expansion in Pseudo-states)

Integral equations:
- Iterative methods (smart starting point)
- Transform into matrix equations
- Multiple scattering expansion
Fig. 1. Q-value diagram for one and two-neutron transfer the system $^6\text{He} + ^{12}\text{C}$. The Q-value for the $^{14}\text{C}$ ground state very positive and introduces a mismatch. For some transitions one- and two-step processes are indicated.
Multichannel definitions

- mass partitions $x$
- spins $I_p$ and $I_t$ and projections $\mu_p$ and $\mu_t$

<table>
<thead>
<tr>
<th>'S basis'</th>
<th>Channel spin $S$</th>
<th>$I_p + I_t = S$</th>
<th>$L + S = J_{\text{tot}}$</th>
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<td>'J basis'</td>
<td>Projectile $J$</td>
<td>$L + I_p = J_p$</td>
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Spin coupling – transforming between LS and JJ

\[ \langle \alpha | \beta \rangle = \sqrt{(2S+1)(2J_p+1)} \, W(L|I_pJ_{tot}I_t; J_pS). \]

Free field limit!

\[ \Psi_{xpt}^{\mu_p\mu_t} (R_x, \xi_p, \xi_t; k_i) \xrightarrow{V=0} e^{ik_i \cdot R_x} \phi_{l_p\mu_p}^{xp} (\xi_p) \phi_{l_t\mu_t}^{xt} (\xi_t) \]
Multichannel wavefunction

- when the interaction is present

\[
\psi_{\mu_{p_i} \mu_{t_i}}^{\mu_{p_i} \mu_{t_i}}(R_x, \xi_p, \xi_t; k_i) = \sum_{J_{tot}M_{tot}} \sum_{\alpha \alpha_i} |\alpha; J_{tot}M_{tot}\rangle \frac{\psi_{\alpha \alpha_i}^{J_{tot}}(R_x)}{R_x} A_{\mu_{p_i} \mu_{t_i}}^{J_{tot}M_{tot}}(\alpha_i; k_i)
\]

where we define an ‘incoming coefficient’

\[
A_{\mu_{p_i} \mu_{t_i}}^{J_{tot}M_{tot}}(\alpha_i; k_i) = \frac{4\pi}{k_i} \sum_{M_i m_i} Y_{L_i}^M(k_i)^* \langle L_i M_i, I_{p_i} \mu_{p_i} | J_{p_i} m_i \rangle \langle J_{p_i} m_i, I_{l_i} \mu_{l_i} | J_{tot} M_{tot} \rangle.
\]

- parity of the full wavefunction

\[
\pi = (-1)^L \pi_{xp} \pi_{xt}
\]
Multichannel S-matrix and T-matrix

- asymptotic behaviour in terms of S-matrix

\[ \psi_{\alpha\alpha_i}^{J_{tot}^\pi} (R_x) = \frac{i}{2} \left[ H_{Li}^-(\eta_\alpha, k_\alpha R_x) \delta_{\alpha\alpha_i} - H_L^+(\eta_\alpha, k_\alpha R_x) S_{\alpha\alpha_i}^{J_{tot}^\pi} \right] \]

- asymptotic behaviour in terms of T-matrix

\[ \psi_{\alpha\alpha_i}^{J_{tot}^\pi} (R_x) = F_{Li} (\eta_\alpha, k_\alpha R_x) \delta_{\alpha\alpha_i} + H_L^+(\eta_\alpha, k_\alpha R_x) T_{\alpha\alpha_i}^{J_{tot}^\pi} \]

\[ S_{\alpha\alpha_i} = \delta_{\alpha\alpha_i} + 2i T_{\alpha\alpha_i} \]
Multichannel coupled equations

\[ H = H_{xp}(\xi_p) + H_{xt}(\xi_t) + \hat{T}_x(R_x) + V_x(R_x, \xi_p, \xi_t) \]

\[ H_{xp}(\xi_p) \phi_{l_p}^{xp}(\xi_p) = \epsilon_{xp} \phi_{l_p}^{xp}(\xi_p), \]

\[ H_{xt}(\xi_t) \phi_{l_t}^{xt}(\xi_t) = \epsilon_{xt} \phi_{l_t}^{xt}(\xi_t), \]

\[ V_x(R_x, \xi_p, \xi_t) = \sum_{i \in p, j \in t} V_{ij}(r_i - r_j) \]

within the same partition, the Schrödinger equations becomes a coupled equation:

\[ [\hat{T}_{xL}(R_x) - E_{xpt}] \psi_{\alpha}(R_x) + \sum_{\alpha'} \hat{V}_{\alpha \alpha'}^{\text{prior}} \psi_{\alpha'}(R_{x'}) = 0. \]
Multi-channel cross section

- For unpolarized beams, we have to sum over final $m$-states and average over initial states:

\[
\sigma_{xpt}(\theta) = \frac{1}{(2I_p + 1)(2I_t + 1)} \sum_{\mu_p \mu_t, \mu_p' \mu_t'} \left| \tilde{f}_{\mu_p \mu_t, \mu_p' \mu_t}(\theta) \right|^2
\]
We can obtain the scattering amplitude in terms of the T-matrix or the S-matrix:

\[ \psi_{\alpha\alpha_i}^{J_{\text{tot}}\pi} (R_x) \overset{R>R_n}{=} H_{L_\alpha}^+ (\eta_\alpha, k_\alpha R_x) T_{\alpha\alpha_i}^{J_{\text{tot}}\pi} \rightarrow i^{-L_\alpha} e^{ik_\alpha R_x} T_{\alpha\alpha_i}^{J_{\text{tot}}\pi} \]

\[ \langle \phi_{lp;\mu_p}^x (\xi_p) \phi_{lt;\mu_t}^x (\xi_t) | \Psi_{xi;\mu_i}^{\mu_i;\mu_t} (R_x, \xi_p, \xi_t; k_i) \rangle \overset{R_x>R_n}{=} f_{\mu_p;\mu_t,\mu_i;\mu_t}^{xpt} (\theta) e^{ik_\alpha R_x / R_x} \]

From the two above equations one can derive

\[ f_{\mu_p;\mu_t,\mu_i;\mu_t}^{xpt} (\theta) = \sum_{J_{\text{tot}}\pi M_{\text{tot}}} \sum_{\alpha\alpha_i} i^{-L_\alpha} \langle \phi_{lp;\mu_p}^x \phi_{lt;\mu_t}^x | \alpha; J_{\text{tot}}M_{\text{tot}} \rangle \times A_{\mu_p;\mu_t,\mu_i;\mu_t}^{J_{\text{tot}}M_{\text{tot}} (\alpha_i; k_i)} T_{\alpha\alpha_i}^{J_{\text{tot}}\pi} , \]

\( \theta \) is the angle between \( k \) and \( k_i \).
Multi-channel scattering amplitude

- plugging in the definition of $A$ and taking into account the Coulomb part:

$$f_{\mu_p\mu_t,\mu_p\mu_t}(\theta) = \delta_{\mu_p\mu_p} \delta_{\mu_t\mu_t} \delta_{\chi_p\chi_p} \chi_i f_c(\theta)$$

$$+ \frac{4\pi}{k_i} \sum_{L_iL_pJ_p,m_im_M;J_{tot}} \langle L_iM_i, I_p\mu_p | J_p m_i \rangle$$

$$= \sum_{L_iL_pJ_p,m_im_M} \langle J_{p_i}m_i, I_{t_i}\mu_{t_i} | J_{tot}M_{tot} \rangle \langle L_M, I_p\mu_p | J_p m \rangle \langle J_p m, I_t\mu_t | J_{tot}M_{tot} \rangle$$

$$= Y_L^M(k) Y_{L_i}^{M_i}(k_i) \tilde{T}^{J_{tot}\pi}_{\alpha\alpha_i}$$

$$= \tilde{T}^{J_{tot}\pi}_{\alpha\alpha_i} = \frac{i}{2} \left[ \delta_{\alpha\alpha_i} - \tilde{S}^{J_{tot}\pi}_{\alpha\alpha_i} \right]$$

- identically one can write the scattering amplitude in LS coupling.
- identically one can write the scattering amplitude in terms of S-matrix.

(3.2.21)
Integrated channel cross section

- channel cross section

\[ \sigma_{xpt}(\theta) = \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum_{\mu_p \mu_{t_1}, \mu_p \mu_{t_1}} \left| \tilde{f}^{xpt}_{\mu_p \mu_1, \mu_p \mu_{t_1}}(\theta) \right|^2 \]

- total outgoing non-elastic cross section

\[
\sigma_{xpt} = 2\pi \int_0^{\pi} d\theta \sin \theta \sigma_{xpt}(\theta)
\]

\[ = \frac{\pi}{k_i^2} \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum_{J_{tot} \pi LJ \alpha_i} (2J_{tot} + 1) |\tilde{S}^{J_{tot} \pi}_{\alpha \alpha_i}|^2 \]

\[ = \frac{\pi}{k_i^2} \sum_{J_{tot} \pi LJ \alpha_i} g_{J_{tot}} |\tilde{S}^{J_{tot} \pi}_{\alpha \alpha_i}|^2 \]
Reaction cross section

- flux leaving the elastic channel (depends only on elastic S-matrix elements)

\[
\sigma_R = \frac{\pi}{k_i^2} \frac{1}{(2I_{p_i}+1)(2I_{t_i}+1)} \sum_{J_{tot} \pi \alpha_i} (2J_{tot}+1)(1 - |S^{J_{tot} \pi}_{\alpha_i \alpha_i}|^2)
\]

\[
= \frac{\pi}{k_i^2} \sum_{J_{tot} \pi \alpha_i} g_J (1 - |S^{J_{tot} \pi}_{\alpha_i \alpha_i}|^2), \text{ similarly.}
\]

- the total cross section is elastic plus reaction cross sections

\[
\sigma_{tot} = \sigma_R + \sigma_{el}
\]

\[
= \frac{2\pi}{k_i^2} \frac{1}{(2I_{p_i}+1)(2I_{t_i}+1)} \sum_{J_{tot} \pi \alpha_i} (2J_{tot}+1)[1 - \text{Re}S^{J_{tot} \pi}_{\alpha_i \alpha_i}]
\]

- absorption cross section

\[
\sigma_A = \sigma_R - \sum_{xpt \neq x_i p_i t_i} \sigma_{xpt}
\]
Absorption cross section

- absorption cross section

\[ \sigma_A = \sigma_R - \sum_{xpt \neq x_ipi} \sigma_{xpt} \]

- the absorption cross section depends on the imaginary part of the optical potential (\(W<0\))

\[ \sigma_A = \frac{2}{\hbar v_i} \frac{4\pi}{k_i^2} \sum_{J_{tot}\pi \alpha \_i \alpha} \int_0^\infty \left[ -W_\alpha (R_x) \right] |\psi_{\alpha \_i \alpha}^{J_{tot}\pi} (R_x)|^2 \, dR_x \]
detailed balance

Consequence of hermiticity: $S$-matrix is unitary

$$\sum_\alpha \tilde{S}_{\alpha\alpha_i}^* \tilde{S}_{\alpha\alpha_i'} = \delta_{\alpha_i\alpha_i'},$$

Even if the $S$-matrix is not unitary, it may be that:

$$|\tilde{S}_{\alpha\alpha_i}|^2 = |\tilde{S}_{\alpha_i\alpha}|^2,$$

Prove that above condition is sufficient for detailed balance:

$$\sigma_{x_ip_it_i:xpt} = \frac{k_i^2 (2I_{p_i} + 1)(2I_t + 1)}{k^2 (2I_p + 1)(2I_t + 1)} \sigma_{xpt:x_ip_it_i}.$$

$$\sigma_{xpt:x_ip_it_i} = \frac{\pi}{k_i^2 (2I_{p_i} + 1)(2I_t + 1)} \sum_{J_{tot}\pi\alpha\alpha_i} (2J_{tot} + 1) |\tilde{S}_{\alpha\alpha_i}^{J_{tot}\pi}|^2.$$
Bare and effective interactions

Effects of neglected direct reaction channels: 2 channel example

\[
[T_1 + U_1 - E_1] \psi_1(R) + V_{12} \psi_2(R) = 0
\]

\[
[T_2 + U_2 - E_2] \psi_2(R) + V_{21} \psi_1(R) = 0
\]

Formally we can solve the second equation and replace it in the first:

\[
[T_1 + U_1 + V_{12} \hat{G}_2^+ V_{21} - E_1] \psi_1(R) = 0.
\]

Where an additional interaction has appeared to account for the effect of the second channel – this interaction is in general non-local and depends on \( E_2 \)

Usually referred to as the dynamic polarization potential

\[
V_{DPP} = V_{12} \hat{G}_2^+ V_{21}
\]

\[
V_{DPP} \psi_1 = V_{12} \hat{G}_2 V_{21} \psi_1
\]

Bare interaction

\[
U_1
\]