Scattering theory: single channel
Including Coulomb
The optical model
Classical Coulomb scattering

- Coulomb trajectories are hyperbolas

\[ \tan \frac{\theta}{2} = \frac{\eta}{bk} \]

- The cross section for a pure Coulomb interaction is

\[ \sigma(\theta) \equiv \frac{b(\theta) \, db}{\sin \theta \, d\theta} = \frac{\eta^2}{4k^2 \sin^4(\theta/2)} \]
Coulomb scattering

- Examples

\[ \frac{d\sigma}{d\Omega} = \left( \frac{Z_1 Z_2 e^2}{4 E_{\text{c.m.}}} \right)^2 \frac{1}{\sin^2 \left( \frac{1}{2} \theta_{\text{c.m.}} \right)} \]

Graphs showing scattering cross-sections for different energy levels.
Coulomb functions

- Coulomb wave equation

\[
\left[ \frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1 \right] X_L(\eta, \rho) = 0
\]

\[
F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} e^{\pm i\rho} \binom{1}{L+1 \mp i\eta; 2L+2; \pm 2i\rho}
\]

\[
C_L(\eta) = \frac{2^L e^{-\pi \eta/2} |\Gamma(1 + L + i\eta)|}{(2L+1)!}
\]

\[
\binom{1}{a; b; z} = 1 + \frac{a}{b} \frac{z}{1!} + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \cdots
\]

\[
H_{L}^{\pm}(\eta, \rho) = G_L(\eta, \rho) \pm iF_L(\eta, \rho)
\]

\[
= e^{\pm i\Theta} (\mp 2i\rho)^{1+L \pm i\eta} U(1+L \pm i\eta, 2L+2, \mp 2i\rho)
\]

\[
\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho)
\]

\[
\sigma_L(\eta) = \arg \Gamma(1 + L + i\eta)
\]
Coulomb functions

- Behaviour near the origin

\[ F_L(\eta, \rho) \sim C_L(\eta) \rho^{L+1}, \quad G_L(\eta, \rho) \sim \left[ (2L+1)C_L(\eta) \rho^L \right]^{-1} \]

\[ C_0(\eta) = \sqrt{\frac{2\pi \eta}{e^{2\pi \eta} - 1}} \quad \text{and} \quad C_L(\eta) = \frac{\sqrt{L^2 + \eta^2}}{L(2L+1)} C_{L-1}(\eta) \]

A transition from small-\(\rho\) power law behavior to large-\(\rho\) oscillatory behavior occurs outside the classical turning point. This point is where \(l = 2\eta/\rho + L(L+1)/\rho^2\), namely

\[ \rho_{tp} = \eta \pm \sqrt{\eta^2 + L(L+1)}. \quad (3.1.67) \]

- Behaviour at large distances

\[ F_L(\eta, \rho) \sim \sin \Theta, \quad G_L(\eta, \rho) \sim \cos \Theta, \quad \text{and} \quad H_L^{\pm}(\eta, \rho) \sim e^{\pm i\Theta} \]

\[ \Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho) \]
Coulomb scattering – partial wave

- pure Coulomb Schrodinger eq can be solved exactly:
  \[ V_c(R) = Z_1 Z_2 e^2 / R \]
  \[ \psi_c(k, R) = e^{ik \cdot R} e^{-\pi \eta/2} \Gamma(1+i\eta) \, _1F_1(-i\eta; 1; i(kR - k \cdot R)) \]

- generalize the partial wave form of the plane wave
  \[ \psi_c(k \hat{z}, R) = \sum_{L=0}^{\infty} (2L+1)i^L P_L(\cos \theta) \frac{1}{kR} F_L(\eta, kR) \]

- asymptotic form of the scattering wavefunction
  \[ \psi_c(k \hat{z}, R) \xrightarrow{R-Z \to \infty} e^{i[kz + \eta \ln k(R-z)]} + f_c(\theta) \frac{e^{i[kR-\eta \ln 2kR]}}{R} \]
Coulomb scattering amplitude

- Formally can be written in partial wave expansion

\[ f_c(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos \theta) (e^{2i\sigma_L(\eta)} - 1) \]

- Series does not converge!

- Without partial wave expansion one can derive the scattering amplitude

\[ f_c(\theta) = \frac{\eta}{2k \sin^2(\theta/2)} \exp \left[ -i \eta \ln(\sin^2(\theta/2)) + 2i \sigma_0(\eta) \right] \]

Point-Coulomb cross section

\[ \sigma_{\text{Ruth}}(\theta) = |f_c(\theta)|^2 = \frac{\eta^2}{4k^2 \sin^4(\theta/2)} \]
Generalized scattering problem w Couomb

- numerical solution is proportional to true solution \( \chi_L(R) = Bu_L(R) \)

\[
\begin{align*}
\psi''(R) = \left[ \frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2} (V(R) - E) \right] \psi(R) \\
B u_L(R) = \chi_L(R) \quad R > R_n
\end{align*}
\]

\[
\chi^{\text{ext}}_L(R) = \frac{i}{2} \left[ \mathcal{H}^-_L(n, kR) - S_L^+ \mathcal{H}^+_L(n, kR) \right]
\]

\[
\begin{align*}
u_L(0) &= 0 \\
u'_L(0) &\neq 0 \\
R_n &> a
\end{align*}
\]
Coulomb+nuclear

- generalized asymptotic form defines the nuclear $S$-matrix

$$\chi^\text{ext}_L(R) = \frac{i}{2} [H^-_L(\eta, kR) - S^n_L H^+_L(\eta, kR)]$$

- can be written in terms of the nuclear phase shift

$$\chi^\text{ext}_L(R) = e^{i\delta^n_L} \left[ \cos \delta^n_L F_L(\eta, kR) + \sin \delta^n_L G_L(\eta, kR) \right]$$

- combined phase shift from Coulomb and nuclear

$$\delta_L = \sigma_L(\eta) + \delta^n_L$$
\[ \delta_L = \sigma_L(\eta) + \delta_L^n \]

\[ e^{2i\delta_L} - 1 = (e^{2i\sigma_L(\eta)} - 1) + e^{2i\sigma_L(\eta)}(e^{2i\delta_L^n} - 1) \]

\[ f_{nc}(\theta) = f_c(\theta) + f_n(\theta) \]

\[ f_n(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1)P_L(\cos \theta) e^{2i\sigma_L(\eta)}(S_L^n - 1) \]
Coulomb+nuclear scattering

$$\sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 \equiv |f_{nc}(\theta)|^2$$

$$\frac{\sigma}{\sigma_{Ruth}} \equiv \frac{\sigma_{nc}(\theta)}{\sigma_{Ruth}(\theta)}$$

Don’t add nuclear only and Coulomb only cross sections!
Optical potential

- Where does the optical potential come from?
  Consider the original many-body problem nucleons-nucleus N+A
  \[ H(r_0; r_1, r_2, \ldots, r_A) \Psi(r_0; r_1, r_2, \ldots, r_A) = E \Psi(r_0; r_1, r_2, \ldots, r_A) \]

Split the Hamiltonian into:
- kinetic energy of the projectile
- the interaction of the projectile with all nucleons of the target
- internal Hamiltonian of the target

\[ H(r_0; r_1, r_2, \ldots, r_A) = T_0 + \sum_{i=1}^{A} V(r_{0i}) + H_A(r_1, r_2, \ldots, r_A) \]

The solutions for the target Hamiltonian form a complete set:
\[ H_A(r_1, r_2, \ldots, r_A) \Phi_i(r_1, r_2, \ldots, r_A) = \epsilon_i \Phi_i(r_1, r_2, \ldots, r_A) \]

The general solution for N+A can be written in terms of the complete set above:
\[ \Psi(r_0; r_1, r_2, \ldots, r_A) = \sum_{ij} \chi_i(r_0) \Phi_j(r_1, r_2, \ldots, r_A) \]
Optical potential

- Feshbach projection
  Since at this point we still assume in our reaction model that the target stays in the ground state, we need to project the problem into the target ground state.

P is the projection operator: \[ P = |\Phi_0\rangle\langle\Phi_0| \]

It picks up the elastic component: \[ P\Psi = \chi_0 \Phi_0 \]

Properties of projection operators

\[ Q = 1 - P \]

\[ P^2\Psi = P\Psi \]
\[ Q^2\Psi = Q\Psi \]
\[ PQ\Psi = QP\Psi = 0 \]

Now apply it to the full equation: \( (E - H)(P + Q)\Psi = 0 \)
Optical potential

\[ |\Psi\rangle = |\Psi_0\rangle + |\Psi_1\rangle + |\Psi_2\rangle \ldots = |\Psi_0\rangle + |\Psi_{in}\rangle \]

\[ H(P + Q)|\Psi\rangle = E(P + Q)|\Psi\rangle \]
\[ PH(P + Q)|\Psi\rangle = PE(P + Q)|\Psi\rangle = EP|\Psi\rangle = E|\Psi_0\rangle \]
\[ QH(P + Q)|\Psi\rangle = QE(P + Q)|\Psi\rangle = EQ|\Psi\rangle = E|\Psi_{in}\rangle \]

\[ H_{PP} = PHP + PVP = T + V_{pp} \]

\[ [E - PHP]|\Psi_0\rangle = PHP|\Psi_{in}\rangle \]
\[ [E - QHQ]|\Psi_{in}\rangle = QHP|\Psi_0\rangle \]
Optical potential

\[
[E - T - V_{PP}]|\Psi_0\rangle = V_{QP}|\Psi_{in}\rangle
\]

\[
[E^{(+)} - T - V_{QQ}]|\Psi_{in}\rangle = V_{QP}|\Psi_0\rangle
\]

\[
|\Psi_{in}\rangle = \left[ E^{(+)} - T - V_{QQ} \right]^{-1} V_{QP} |\Psi_0\rangle
\]

\[
H_{PP} = PHP = T + PVP = T + V_{pp}
\]

\[
H_{QQ} = QHQ = T + QVQ = T = V_{QQ}
\]

\[
[H_{PP} + V_{PP} + [E^{(+)} - T - V_{QQ}]^{-1} V_{QP} |\Psi_0\rangle = 0
\]

\[
V_{PP}^{opt} = V_{PP} + V_{PQ} [E^{(+)} - T - V_{QQ}]^{-1} V_{QP}
\]
Optical potential

- Now making it explicit:

\[
\left\{ E - T_0 - \langle \Phi_0 | V | \Phi_0 \rangle - \langle \Phi_0 | V Q \frac{1}{E - QHQ} QV | \Phi_0 \rangle \right\} \chi_0 = 0
\]

\[
V \equiv \sum_{i=1}^{A} V(r_{0i})
\]

Potential acting between projectile and target nucleons

Interpretation for the formal propagator: multiple scattering in Q-space

\[
\frac{1}{E - QHQ} = \frac{1}{E} \left\{ 1 + \frac{1}{E} QHQ + \frac{1}{E} QHQ \frac{1}{E} QHQ + \cdots \right\}
\]

- The scattering equation can be rewritten:

\[
(E - T_0 - V(r_0)) \chi_0 = 0
\]

with the effective potential:

\[
V(r_0) = \langle \Phi_0 | V | \Phi_0 \rangle + \langle \Phi_0 | V Q \frac{1}{E - QHQ} QV | \Phi_0 \rangle
\]

Wong, Introduction to Nuclear Physics, Wiley
Optical potential

- The scattering equation can be rewritten: 
  \[(E - T_0 - \mathcal{V}(r_0))\chi_0 = 0\]
  with the effective potential:
  \[\mathcal{V}(r_0) = \langle \Phi_0|V|\Phi_0 \rangle + \langle \Phi_0|VQ\frac{1}{E - QHQ}QV|\Phi_0 \rangle\]

  Deriving this is part of homework 6

- This potential is generally non-local which gives rise to some complications:
  \[(E - T_0)\chi_0(r_0) = \mathcal{V}(r_0)\chi_0(r_0) + \int f(r_0, r'_0)\chi_0(r'_0)dr'_0\]

  Often this is approximated to a local version.
  The optical model replaces this microscopic potential by a model potential obtained phenomenologically:
  \[(E - T_0 - U_{\text{opt}})\chi_0 = 0\]

  Scattering into Q-space may never return to elastic – loss of flux
  Optical potential needs to have an imaginary term!

Wong, Introduction to Nuclear Physics, Wiley
Optical potential

\[ \psi(x) = e^{ikx} \]

\[ \bar{\psi}(x) = e^{i\bar{k}x} \]

\[ k^2 = \frac{2\mu}{\hbar^2} (E + V_0) \]

\[ \bar{k}^2 = \frac{2\mu}{\hbar^2} (E + V_0 + iW_0) \]

\[ \bar{k}^2 = \frac{2\mu}{\hbar^2} (E + V_0 + iW_0) = \frac{2\mu}{\hbar^2} (E + V_0) \left[ 1 + \frac{iW_0}{E + V_0} \right] \]

\[ \bar{k} = k \left[ 1 + \frac{iW_0}{E + V_0} \right]^{1/2} \approx k \left[ 1 + \frac{iW_0}{2(E + V_0)} \right], \quad W_0 \ll E, V_0 \]

So, for \( W_0 > 0 \), \( \bar{k} = k + ik_i/2, \quad k_i = kW_0/(E + V_0) > 0 \),

\[ \bar{\psi}(x) = e^{i\bar{k}x} = e^{ikx} e^{-\frac{1}{2}k_i x}, \quad |\bar{\psi}(x)|^2 = e^{-k_i x} \]
Optical potential

- Example of a microscopically derived optical potential: folding

\[ U_{\text{opt}}(r_0) \approx \langle \Phi_0(r_1, r_2, \ldots, r_A) | \sum_{i=1}^{A} V(r_i) | \Phi_0(r_1, r_2, \ldots, r_A) \rangle \]

free or medium NN interaction? density dep?

- In principle antisymmetrization need to be included:

\[ \langle \Phi_0 | V | \Phi_0 \rangle = \langle \Phi_0 | t_D | \Phi_0 \rangle + \langle \Phi_0 | t_E | \Phi_0 \rangle \]

direct \hspace{1cm} exchange

Wong, Introduction to Nuclear Physics, Wiley
Optical potential

In principle antisymmetrization need to be included:

\[ \langle \Phi_0 | V | \Phi_0 \rangle = \langle \Phi_0 | t_D | \Phi_0 \rangle + \langle \Phi_0 | t_E | \Phi_0 \rangle \]

Direct part depends of the density:

\[ U_{opt}^D (r_0, E) = \int \rho(r) t_D (r_0, r, \rho, E) \, d^3 r \]

\[ \rho(r) = |\Phi_0 \rangle \langle \Phi_0 | \approx \sum_{i=1}^{A} \phi_i^* (r) \phi_i (r) \]

The exchange part is non-local in general

Radial shape for the real part of p+A optical potential at different beam energies: NN-Paris potential

*Wong, Introduction to Nuclear Physics, Wiley*
Optical potentials

- all the terms to be considered: 
  \[ V_c(R) + V(R) + i \, W(R) + V_{so}(R) \]
  - loss of flux - absorption \((W<0)\)

- Nucleon potentials as described with Woods-Saxon shape
  (to mimic the density distribution in nuclei)

\[
V(R) = -\frac{V_r}{1 + \exp\left(\frac{R-R_r}{a_r}\right)}
\]
\[
W(R) = -\frac{W_i}{1 + \exp\left(\frac{R-R_i}{a_i}\right)}
\]

- Sometimes imaginary also defined at \(d/dR(V_{ws}(r))\) - surface

For nucleon interaction \(V=40-50\text{ MeV}, r=1.2\text{ fm and }a=0.6-0.65\text{ fm}\)
reaction and absorptive cross section

$$\chi^\text{ext}_L(R) = \frac{i}{2} [H^-_L(\eta, kR) - S^n_L H^+_L(\eta, kR)]$$

Single channel: $|S_L|^2$
Probability of elastic scattering

Reaction cross section: everything but elastic

$$\sigma_R = \frac{\pi}{k^2} \sum_L (2L+1)(1 - |S_L|^2)$$

Absorption cross section: everything that was lost due to the imaginary potential

$$\sigma_A = \frac{2}{\hbar \nu} \frac{4\pi}{k^2} \sum_L (2L + 1) \int_0^\infty [-W(R)] |\chi_L(R)|^2 \, dR$$

For simple spherical potentials, reaction and absorptive cross sections are the same
Rutherford scattering: examples

- Purely Coulomb potential ($\eta \gg 1$)
- Bombarding energy well below the Coulomb barrier
- Obeys Rutherford law:

\[
\frac{d\sigma}{d\Omega} = \frac{zZe^2}{4E} \frac{1}{\sin^4(\theta/2)}
\]

Courtesy of Antonio Moro
elasticsearch: examples

**FRAUNHOFER SCATTERING:**

- Bombarding energy well above Coulomb barrier
- Coulomb weak ($\eta \lesssim 1$)
- Nearside/farside interference pattern (diffraction)
elastic scattering: examples

FRESNEL SCATTERING:

- Bombarding energy around or near the Coulomb barrier
- Coulomb strong ($\eta \gg 1$)
- 'Illuminated' region $\Rightarrow$ interference pattern (near-side/far-side)
  - 'Shadow' region $\Rightarrow$ strong absorption
elastic scattering: examples

How does the halo structure affect the elastic scattering?

- $^4\text{He}+^{208}\text{Pb}$ shows typical Fresnel pattern → **strong absorption**
- $^6\text{He}+^{208}\text{Pb}$ shows a prominent reduction in the elastic cross section due to the flux going to other channels (mainly break-up)
- $^6\text{He}+^{208}\text{Pb}$ requires a large imaginary diffuseness → **long-range absorption**
In Fraunhofer scattering the presence of the continuum produces a reduction of the elastic cross section.