1 Transverse Dynamics

Transverse dynamics of accelerator focuses on the dynamics of transporting the charged particle to a specific location (transpo condition) a accelerator ring.

![Image 1](transverse_dynamics.png)

To affect the motion of a charged particle, we need external magnetic field or electric field, followed by the Lorentz force:

\[ \mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

1.1 Magnetic or Electric Fields?

To have the same Lorentz force on the charged particle, the comparable magnetic field and electric field is scaled by velocity of electric field need to have same force of 1 Tesla magnetic field.

<table>
<thead>
<tr>
<th>$\gamma - 1$</th>
<th>$\frac{F_0}{F_e}$</th>
<th>$\beta c$</th>
<th>$B$ (T)</th>
<th>$E$ (MV/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-5}$</td>
<td>0.0045c</td>
<td>1</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>0.0141c</td>
<td>1</td>
<td>4.24</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.0447c</td>
<td>1</td>
<td>13.4</td>
<td></td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>0.140c</td>
<td>1</td>
<td>42.1</td>
<td></td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>0.417c</td>
<td>1</td>
<td>125</td>
<td></td>
</tr>
</tbody>
</table>

Considering the field break down limit of the electric field, we see that only when the particle velocity is not relativistic, the electric field is widely used to bend/steer/focus the relativistic particles.

1.2 Design Orbit and Coordinate System
The geometry of the accelerator is determined by the dipole magnet. Let us assume that the magnetic field is constant in certain direction \( \mathbf{B} = B_0 \mathbf{\hat{y}} \). The particle travels through this region initially in \( \mathbf{\hat{x}} \) direction in the \( x-y \) plane, as shown below.

Note that the coordinate moves along the ideal beam trajectory and the \( \mathbf{\hat{z}} \) is always along the tangent direction and \( \mathbf{\hat{r}} \) is pointing named Frenet-Serret coordinate system.

The equation of motion is

\[
\frac{d\mathbf{p}}{dt} = q \mathbf{v} \times \mathbf{B}
\]

The momentum change will be only in the \( x-y \) plane, the velocity is always perpendicular to the magnetic field and the energy \( E \)

\[
E = \frac{1}{2} m \mathbf{v}^2 = q B_0 \mathbf{v} \times \mathbf{\hat{y}}
\]

Using the fact that the amplitude of the velocity will not change therefore:

\[
\frac{m \mathbf{v}^2}{\rho} = q \mathbf{v} B_0
\]

The bending radius is given by:

\[
\rho = \frac{|\mathbf{p}|}{q B_0}
\]

Here, we define the term ‘rigidity’ \( B_0 \rho = \frac{1}{q E} \), to quantify ‘how hard to bend the beam’. And naturally it is only function of the \( p \) very important normalization factor of the transverse motion, as will shown later. After normalization, most our calculation becor

The dipoles that defines the geometry of the accelerator are sometimes called main dipoles. They define a ideal trajectory. A ch inject with correct position and angle, will exactly follow the ideal trajectory.

1.3 Magnetic Multi-poles

There are many common type of magnets used accelerators, they are grouped using the order of multi-pole expansion in the vi where the particles is transported. Here are the list of common type of magnets

<table>
<thead>
<tr>
<th>Magnet Types</th>
<th>Usages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipoles</td>
<td>Form the Geometry of Accelerator</td>
</tr>
<tr>
<td>Quadrupoles</td>
<td>Focus/Defocus the beam</td>
</tr>
<tr>
<td>Sextupole</td>
<td>Chromatic correction, nonlinear effects</td>
</tr>
<tr>
<td>Octopole</td>
<td>Damping of the collective effects</td>
</tr>
<tr>
<td>Solenoid</td>
<td>Focusing for low energy beam</td>
</tr>
<tr>
<td>Correctors</td>
<td>Steer the beam</td>
</tr>
</tbody>
</table>

In a vacuum chamber, the region is sourceless, i.e. there is no charge or current. The magnet field can be expressed using ‘B’

\[
B = B_0 \sum_{n=0}^{\infty} (b_n + i a_n) (x + iy)^n
\]

(U.S. convention)

\[
= B_0 \sum_{m=1}^{\infty} (b_m + i a_m) (x + iy)^{m-1}
\]

(European convention)

Here, \( B_0 \) is the main dipole field strength. The \( b_n \) and \( a_n \) are called the \( 2(n + 1)^{th} \) multipole coefficients in ‘U.S. convention’ (st \( a_m \) are called \( 2m^{th} \) multipole coefficients in ‘European convention’ (starting from \( m = 1 \). The set of \( b \) coefficients are the norm coefficients are the skew components.

https://people.nscl.msu.edu/~haoy/teaching/fundamental_AP/transverse_dynamics/transverse_dynamics.html
To achieve high field, accelerator usually use high permeability material to boost the flux density $B$ for the same magnetic field $H$:

$$B = \mu \mu_0 H$$

The figure below shows the permeability of ferromagnet material as function of the external magnetic field $H$:

![Permeability of ferromagnetic material](image)

Fig.4 - Permeability of the ferromagnetic material

In the magnet design, we use iron cores as the ferromagnet materials. Iron (99.8% pure) may have initial $\mu_r$ of 150 and can reach pure iron will have initial $\mu_r$ of 10000 and reach maximum 200000.

At such high permeability, the magnetic field line is always perpendicular to the surface of the iron core.

### 1.3.1 Dipole
If only the lowest order of coefficients are non-zero values, the field reads:

\[
B_x = B_0 b_0 \\
B_z = B_0 a_0
\]

in US convention.

The main dipoles will have \( b_0 = 1 \) and \( a_0 = 0 \). Below, two types of dipoles magnet are shown in the cartoon:

Since the dipole bend the ideal trajectory. The both edges of the dipoles may affect the motion of the particle. Below, dipole wit most useful types are sector dipoles, with zero edge angles on each side and rectangular dipoles who has both edge angles eq

1.3.2 Quadrupole

The magnet is named quadrupole when the quad coefficient \( b_1 \) and/or \( a_1 \) are non-zero (US convention). They are named normal accordingly. The field line of the quadrupole are:
The normal quadrupole has the magnetic field as:

\[ \mathbf{B} = B_0 b_1 (y\hat{x} + x\hat{y}) \]

while the skew quadrupole has the magnetic field as:

\[ \mathbf{B} = B_0 a_1 (x\hat{x} - y\hat{y}) \]

### 1.3.3 Sextupole

The magnet is named Sextupole when the sextupole coefficient \( b_3 \) and/or \( a_3 \) are non-zero (US convention). They are named \( r \) accordingly. The field line of the sextupole are:

![Normal Sextupole](image1.png)

![Skew Sextupole](image2.png)

The normal sextupole has the magnetic field as:

\[ \mathbf{B} = B_0 b_2 \left[ 2xy\hat{x} + (x^2 - y^2)\hat{z} \right] \]

while the skew Sextupole has the magnetic field as:

\[ \mathbf{B} = B_0 a_2 \left[ (x^2 - y^2)\hat{x} - 2xy\hat{z} \right] \]

### 1.4 Linear Transverse Motion

#### 1.4.1 Motion in Quadrupole

The simplest motion of the charged particle is in normal quadrupole, whose field is

\[ B_x = G_y \]
\[ B_y = G_x \]

where the gradient is \( G = B_0 b_1 \).

Quadrupole is usually positioned so that the zero field axis is located on the ideal trajectory. The ideal particle will not experienc...
From the Lorentz force, we have
\[
\frac{dy}{dt} = qv \times B
\]
Expand the vector production and get
\[
\frac{dy}{dt} = -qGv_y = qGv_z
\]
Here we need to make following approximation:

- **paraxial approximation**, indicate that the particle will only deviate the design direction by small angles, i.e. if \( z \) is the forwa velocity satisfy: \( v_x, v_y \ll |v| \)

- In such approximation,
\[
v_z = \sqrt{v_x^2 - v_y^2 - v_z^2}
\]
\[
v_z = v \left( 1 - \frac{v_x^2 + v_y^2}{2v_z^2} \right) \sim v
\]

- Using such approximation, we have
\[
\frac{dv}{dt} = v \frac{dv}{ds} \sim v \frac{d^2y}{ds^2} \sim v^2 \frac{d^3y}{ds^3}
\]

Then the equation of motion in quadrupole is simplified as
\[
\frac{ymv}{ds^2} = qGx
\]
\[
\frac{ymv}{ds^2} = -qGy
\]

Using the definition of the rigidity \( B \rho \), the EOM can be rewritten as:
\[
\frac{d^2x}{ds^2} = -\frac{G}{B \rho} x
\]
\[
\frac{d^2y}{ds^2} = \frac{G}{B \rho} y
\]

Then, we define the normalized quadrupole strength \( k = G(B \rho) \), the EOM is simply:
\[
\frac{d^2x}{ds^2} + k x = 0
\]
\[
\frac{d^2y}{ds^2} - ky = 0
\]

In later syntax, we use prime (') to denote the derivative with respective to \( s \).

Depend on the sign of \( k \), it is easy to solve the differential equation:
\[
x(s) = \begin{cases} 
  a \cos(\sqrt{k} s) + b \sin(\sqrt{k} s) & k > 0 \\
  as + b & k = 0 \\
  a \cosh(\sqrt{-k} s) + b \sinh(\sqrt{-k} s) & k < 0
\end{cases}
\]

The three cases corresponds to a focusing quadrupole, drift space and a defocusing quadrupole respectively. The parameter \( a \), position \( x \) and its derivative \( x' \). We also learn that the quadrupole will always focus the particles in one transverse direction and

### 1.4.2 Motion in Dipole
Dipole itself will not affect the beam other than bending. However, the choice of the Frenet–Serret system results a focusing effect.

If the particle is injected $x (>0)$ away from the ideal trajectory, the particle will travel longer in the dipole than the ideal particle, with the bending angle $\theta$ is small, we have:

$$\Delta x' = -\frac{x}{\rho} \theta = -\frac{x}{\rho^2} \Delta s$$

$$\Delta y' = 0$$

Therefore the particle satisfy the following differential equation in dipole:

$$x'' + \frac{1}{\rho^2} x = 0$$

$$y'' = 0$$

And the solution of the equation of motion is simply:

$$x(s) = a \cos \left( \frac{s}{\rho} \right) + b \sin \left( \frac{s}{\rho} \right)$$

$$y(s) = as + b$$

### 1.4.3 Hill's Equation

Let’s combine the effects of the dipole and quadrupole and write the equation of motion as the form of Hill’s equation:

$$x'' + \left( \frac{1}{\rho^2(s)} + k(s) \right) x = 0$$

$$y'' - k(s)y = 0$$

In general, the radius $\rho$ and focusing strength $k$ are functions of the longitudinal coordinate $s$. If the parameter is piecewise constant last two sections.

### 1.5 Transverse Matrix

It is convenient to write the solution of the Hill’s equation using matrix form. Such matrix is named betatron transfer matrix. The passing through a specific magnet can be described by $2 \times 2$ matrix $M$,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

as:

$$\begin{pmatrix} x' \\ x'' \end{pmatrix}_{exit} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{entrance}$$

### 1.5.1 Drift space
The EOM for drift space is simply $x'' = 0$ and $y'' = 0$. Using horizontal direction as instance, if the initial condition at the start $x'(s = s_0) = x'_0$. The solution is:

$$x(s = s_0 + l) = x_0 + x'_0 l$$

Or in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_m = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_m$$

We write the transfer matrix for drift space as:

$$M(s_0 + l, s_0) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

### 1.5.2 Dipoles

For the dipole that bend the beam with the radius $\rho$ in horizontal plane, the $x$ direction follows an EOM as a harmonic oscillator, as in a drift space.

The matrix form in the dipole can be easily get as:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_m = \begin{pmatrix} \cos(\frac{l}{\rho}) & \rho \sin(\frac{l}{\rho}) \\ -\sin(\frac{l}{\rho})/\rho & \cos(\frac{l}{\rho}) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_m$$

The transfer matrix for dipole in horizontal space is:

$$M(s_0 + l, s_0) = \begin{pmatrix} \cos(\frac{l}{\rho}) & \rho \sin(\frac{l}{\rho}) \\ -\frac{1}{\rho} \sin(\frac{l}{\rho}) & \cos(\frac{l}{\rho}) \end{pmatrix}$$

In a small angle approximation $l/\rho \ll 1$, we can expand the matrix up to first order of $l/\rho$. The matrix reduce to a drift space:

$$M(s_0 + l, s_0) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

### 1.5.3 Quadrupoles

In the example before, we already derived the solution for particle in the quadrupole. It can be rewritten in the matrix form as

$$M(s_0 + s, s_0) = \begin{pmatrix} \cos(\sqrt{k}s) & \sin(\sqrt{k}s)/\sqrt{k} \\ -\sqrt{k} \sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \quad k > 0, \text{Focusing Quad}$$

$$M(s_0 + s, s_0) = \begin{pmatrix} \cosh(\sqrt{k}s) & \sinh(\sqrt{k}s)/\sqrt{k} \\ \sqrt{k} \sinh(\sqrt{k}s) & \cosh(\sqrt{k}s) \end{pmatrix} \quad k < 0, \text{Defocusing Quad}$$

Usually, the quadrupole has short length and strong field gradient. We may model such short quad as a ‘lens with zero length’, i.e.

$$f = \lim_{l \to 0} \frac{1}{|k|}$$

In such limit the transfer map for a thin-length quad is

$$M_{quad} = \begin{pmatrix} 1 & 0 \\ -lf & 1 \end{pmatrix} \quad \text{Focusing Quad}$$

$$M_{quad} = \begin{pmatrix} 1 & 0 \\ lf & 1 \end{pmatrix} \quad \text{Defocusing Quad}$$

### 1.5.4 Chain of linear elements
In accelerators, we have chain of magnets, as illustrated above. Each magnet has a transfer matrix, the transfer matrix for the \(c\) multiplications:

\[
M(x_n, x_0) = M(x_n, x_{n-1}) \cdots M(x_2, x_1) M(x_1, x_0)
\]

Note that the order of the multiplication is the reverse order of the placement of the magnet.

Usually we call the sequence of the magnets as **lattice**.

### 1.5.5 Properties of the transverse matrix

#### 1.5.5.1 Symplectic condition

The matrix represents the dynamics of charged particle, which obeys the energy conservation law. We claim the property of the **symplectic condition**, in \(2 \times 2\) matrix case, the symplectic condition becomes:

\[
\det(M) = 1
\]

#### 1.5.5.2 Long-term stability condition

The symplecticity ensures the area phase space is constant. However, in most times, we have stronger requirements for a repeller.

In accelerator, the beam usually will pass through a chain of elements (called cell) many times. A long lattice usually consist of \(n\) particle will travels through the lattice billions of turns. If the transfer matrix for one cell is \(M\), we are interested in the behavior of

\[
M_{\text{total}} = M \cdot M \cdots M = M^k
\]

where \(k\) is a very large integer or infinity.

To avoid the either \(x\) or \(x^*\) to be very large (or infinity) after \(k\) cells. We required that the absolute value of the eigenvalues of \(M\) be found by:

\[
\begin{vmatrix}
\lambda - a & -b \\
-c & \lambda - d
\end{vmatrix} = 0
\]

or

\[
\lambda^2 - (a + d)\lambda + 1 = 0
\]

To satisfy \(|\lambda| \leq 1\), we require

\[
|\text{Tr}(M)| = |a + d| \leq 2
\]

This is the stable condition for one cell in periodical structure.

### 1.6 Twiss Parametrization
To separate description of transverse motion from contribution the lattice (magnet arrangement) and the contribution from the p
parametization.

For a periodical lattice, the matrix is written as:

$$M = \begin{pmatrix}
\cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\
-\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi
\end{pmatrix}$$

Here, \(\beta(s), \alpha(s)\) and \(\gamma(s)\) are twiss functions that only depend on the lattice. \(\Phi\) is the phase advance of the periodical lattice. \(T\)
\(\beta = 1 + \alpha^2\)

The unit of beta function is unit of length and alpha function is unitless.

The particle's transverse position at the \((n+1)\)th can be found by:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_n$$

If we know the beta function of every location in the lattice, the motion of the particle is written as:

$$x(s) = \sqrt{2T\beta(s)} \cos \psi(s)$$

\(J\) is called action of the particle, which is only depend the initial condition of the particle. \(\psi\) is the phase advance from the origi
coordinate \((x, x')\) under the transfer map using \(\beta = 4\pi, \alpha = -2\) and \(\psi(2\pi) = 0.2389\).

As seen in the below example, the beta and alpha function determines the shape (orientation) of the ellipse that the particle trac

In [4]:

```python
from map2D import map2D
import numpy as np
beta=4
alpha=-2
tunex=0.2389
turns=1000

xp=map2D(npard=1, twiss=[beta, alpha], tune=tunex, chrom=0.0, espr=0.0,
        particles=np.vstack([np.array([[0.0008],[0,0]])])
stats=xpx.track(turns)

import matplotlib.pyplot as plt
%matplotlib notebook
fig,ax=plt.subplots()
ax.set_xlabel('x [mm]')
ax.set_ylabel('x' [mrad]')
lines=ax.plot(stats[0]*le3,stats[1]*le3, linestyle='-', marker='.', markersize=0
#ax.plot(stats[2]*le3,stats[3]*le3, linestyle=':', marker='*', markersize=0
fig.savefig('ellipse.png', bbox_inches='tight')
```
In [18]:

```python
In[18]:
%matplotlib inline
from matplotlib.animation import FuncAnimation

fig, ax = plt.subplots()
fig, ax = plt.subplots()
ax.set_xlabel(r'$x$ [mm]')
ax.set_ylabel(r'$x'$ [mrad])
lines, = ax.plot(stats[:, 0] * 1e3, stats[:, 1] * 1e3, linestyle='-', marker='.', markersize=0)
line, = ax.plot([], [], linestyle='-', marker='+', markersize=12, color='g')
def init():
    line.set_data(stats[0, 0] * 1e3, stats[0, 1] * 1e3)
    return line,
def run(i):
    # update the data
    #result = next(evolve)
    line.set_data(stats[i, 0] * 1e3, stats[i, 1] * 1e3)
    return line,

anim = FuncAnimation(fig, run, frames=100, interval=500,)
from IPython.display import HTML
HTML(anim.to_html5_video())
```

Out[18]:

![Plot](image_url)

If we get the $2 \times 2$ transfer map of one location $x_0$ in a periodic lattice structure:

$$M(x_0) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

It should be symplectic or the determinant is one for 1-D case. The optical function and phase advance of one period can be $e^{i\phi}$

Then, optical functions of another location can be calculated from its one turn matrix:

$$M(x_1) = M(x_1 \mid x_0)M(x_0 \mid x_0)$$

where $M(x_1 \mid x_0)$ represents the transfer map from location $x_0$ to $x_1$. 
If we know the optical function of two locations and the phase advance between them, the transfer function is given as:

\[
M(s_1 | s_0) = \begin{pmatrix}
\sqrt{p_2} \cos \psi + \alpha_0 \sin \psi & \sqrt{p_2} \sin \psi \\
-\frac{1}{\sqrt{p_2}} \sin \psi + \frac{\alpha_0}{\sqrt{p_2}} \cos \psi & \sqrt{p_2} \cos \psi - \alpha_1 \sin \psi
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\sqrt{p_2}} & 0 \\
-\frac{\alpha_1}{\sqrt{p_2}} & \sqrt{p_2}
\end{pmatrix}^{-1}
\begin{pmatrix}
\cos \psi & \sin \psi \\
-\sin \psi & \cos \psi
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{p_2}} & 0 \\
\frac{\alpha_0}{\sqrt{p_2}} & \sqrt{p_2}
\end{pmatrix}
\]

where \( \psi \) is the phase advance between \( s_0 \) and \( s_1 \).

### 1.7 Example: FODO cell

FODO cell is a type of widely used cell of magnets. It consist of only two quadrupoles and two dipoles or simply drifts between use the following sequence:

\[
\frac{1}{2} QF - O - QD - O - \frac{1}{2} QF
\]

We use the following approximation:

- Small bending angle approximation: the transfer matrix for a dipole reduces to a drift space, the space between quads are short quadruple approximation, so that the quad can be modeled by its focusing length \( f \)
- Focusing length and defocusing length are same.

Then the matrix is

\[
M_{\text{FODO}} = \begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\begin{pmatrix}
L_0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix}
\begin{pmatrix}
1 & L_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\]

In [5]:

```python
import sympy
sympy.init_printing(use_unicode=True)

l,f,a=sympy.symbols("L_0, f, alpha")
def mat_q(f):
    return sympy.Matrix([[1,0],[1/f,1]])
def mat_d(d):
    return sympy.Matrix([[1,d],[0,1]])

mat_fodo=sympy.simplify(mat_q(-2*f)*mat_d(l)*mat_q(f)*mat_d(l)*mat_q(-2*f))
mat_fodo
```

Out[5]:

\[
\begin{pmatrix}
\frac{L_0}{f} + \frac{L_0 (l_0 + 2f)}{f} + 1 \\
\frac{L_0 (l_0 - 2f)}{f} & -\frac{L_0}{f} + 1
\end{pmatrix}
\]

The transfer matrix to/from the mid-point of the focusing quad is

\[
M_{\text{FODO}} = \begin{pmatrix}
-\frac{L_0}{f} + 1 & \frac{L_0 f (L_0 + 2f)}{f} \\
\frac{L_0 f (L_0 - 2f)}{f} & -\frac{L_0}{f} + 1
\end{pmatrix}
\]

The determinant of the matrix is always 1. The trace of the FODO cell is \( 2 - \frac{L_0^2}{f^2} \).

Then the betatron phase advance per cell can be calculated from:

\[
\cos \Phi = \frac{1}{2} \text{tr}(M) = 1 - \frac{L_0^2}{2 f^2}
\]

or

\[
\sin \Phi = \frac{L_0}{2 f}
\]

And the beta and alpha function at the mid-point of the focusing quad is:

\[
\beta_F = \frac{2L_0 (1 + \sin(\Phi/2))}{\sin \Phi}
\]

\[
\alpha_F = 0
\]
1.8 Emittance of particles

We can plot all particles’ location and momentum \((x,x')\) in the phase space. The area of the phase space that the ensemble of term ‘emittance’. We can use the same example as above to illustrate:

```python
In [5]: from map2D import map2D, phase_ellipse
   import numpy as np
beta=4
alpha=-2
emit=1e-6
tunex=0.2389
turns=1

xpx=map2D(npart=10000, emit=emit, twiss=[beta, alpha], tune=tunex, chrom=0.0, espr=0.
x_rms.px_rms=phase_ellipse(beta, alpha, emit)
x_6rms.px_6rms=phase ellipse(beta, alpha, 6*emit)
import matplotlib.pyplot as plt
%matplotlib notebook
fig, ax=plt.subplots()
ax.set_xlabel(r'$x$ [mm]
ax.set_ylabel(r'"x' [mrad]
ax.plot(xpx.coor2D[0]*1e3,xpx.coor2D[1]*1e3, linestyle='', marker='.', markersize=0.
ax.plot(x_rms*1e3, px_rms*1e3)
ax.plot(x_6rms*1e3, px_6rms*1e3)
```

![Image of phase space plot]

Out[5]: [<matplotlib.lines.Line2D at 0x109d3f1d0>]

The **rms emittance** is calculated as:

\[
\epsilon_{\text{rms}} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{x'x}^2}
\]

The portion of particle in the emittance ellipse is given by:

<table>
<thead>
<tr>
<th>ratio (c/\epsilon_{\text{rms}})</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>63%</td>
</tr>
<tr>
<td>4</td>
<td>86%</td>
</tr>
<tr>
<td>6</td>
<td>95%</td>
</tr>
</tbody>
</table>

1.9 Dispersion and Chromaticity

https://people.nscl.msu.edu/~haoy/teaching/fundamental_AP/transverse_dynamics/transverse_dynamics.html
When particle has different energy (called off-momentum particle) than reference energy (correspond to the rigidity $B\rho$), two effects are considered.

First, the dipole will bend the particle differently, as shown in the below figure:

The corresponding Hill's equation reads:

$$x'' + \left( \frac{1}{\rho^2} + k(s) \right) x = \frac{\delta}{\rho}$$

We define the dispersion assuming $x = x_0 + D\delta$, with $\delta$ as a energy parameter $\delta = (P - P_0)/P_0$. Then the dispersion is defined as:

$$D(s) = \frac{dx(s)}{d\delta}$$

which satisfy the same inhomogeneous differential equation:

$$D'' + \left( \frac{1}{\rho^2} + k(s) \right) D = \frac{1}{\rho}$$

The calculation of the dispersion will be covered in accelerator physics courses.

Another effect for an off-momentum particle is that the focal length of the quadrupole changes. Consequently, the phase advance of the particle changes. We will define the linear chromaticity $\xi$ as:

$$\xi = \frac{d\phi}{d\delta}$$

Both dispersion and chromaticity are inevitable since there is always dipoles and quadrupoles in the lattice. Many beam dynamics control of dispersion and chromaticity is essential component in accelerator design.