FRIB

Electron Synchrotrons

Steve Lidia

MICHIGAN STATE UNIVERSITY

U.S. DEPARTMENT OF ENERGY | Office of Science

This material is based upon work supported by the U.S. Department of Energy Office of Science under Cooperative Agreement DE-SC0000661, the State of Michigan and Michigan State University. Michigan State University designs and establishes FRIB as a DOE Office of Science National User Facility in support of the mission of the Office of Nuclear Physics.

Outline

- History and examples
- Accelerator model review
- Linear and Nonlinear resonances
- Synchrotron radiation
- Lifetimes
- (Appendixes)

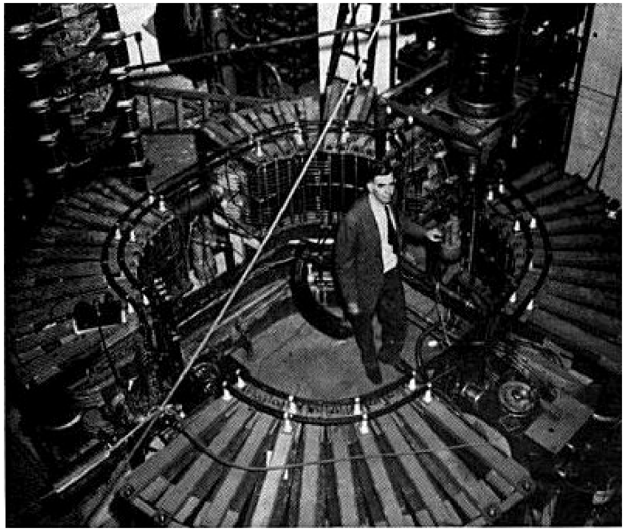
Introduction:
Electron Rings

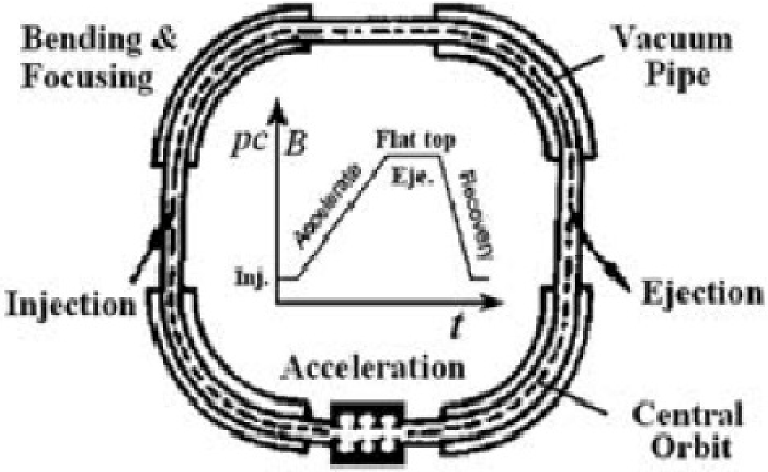
- A ring is a an electromagnetic system with a closed particle orbit.
  - The closed orbit is a natural choice of the reference orbit in rings. The motion of particles typically is described relatively to the closed orbit.
- We will be interested in systems with a stable orbit. That is, particles with a small enough deviation from the closed orbit are stable in respect to the closed orbit.
- Electrons in circular accelerators can make many turns and interact with accelerating RF many times, reaching high energy over an extended period of time.
  - In linacs, this happens only once or several times (recirculating linacs).
- Also, rings can store electrons (and positrons) for significant amount of time (hours), providing unique experimental capabilities as colliders and synchrotron light sources.

Introduction: Electron Synchrotron Boosters

- Electron synchrotron boosters accelerate electrons to a specific energy to inject them into other accelerators. Electron linacs are frequently used as injectors to boosters: source→linac→booster ring→storage ring
- Historically, boosters were used for fixed target experiments. However, those machines have been decommissioned long time ago

The first synchrotron to use the "racetrack" design with straight sections, a 300 MeV electron synchrotron at University of Michigan in 1949, designed by Dick Crane.

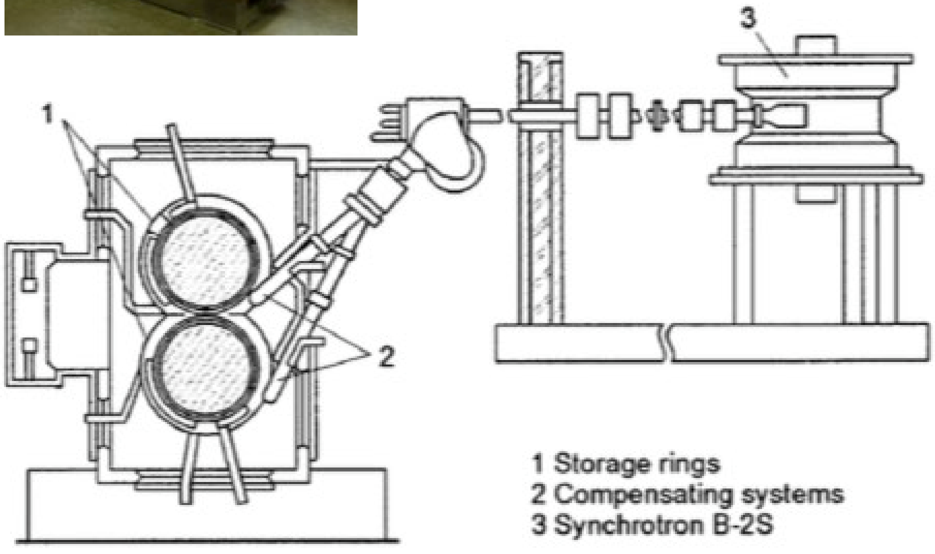




Introduction:
Electron-Electron, Electron-Positron Colliders

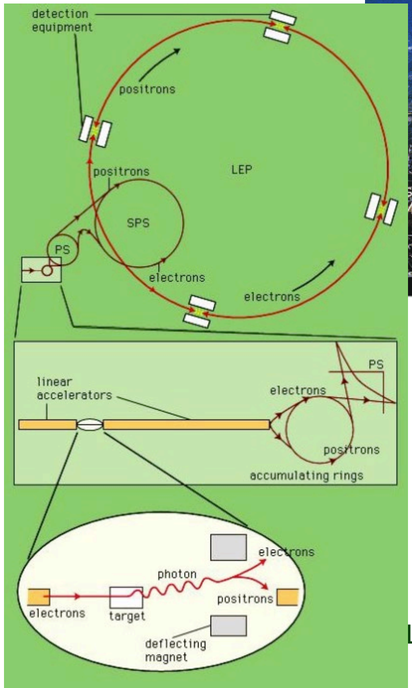
VEP-1, 1963
Russia, Novosibirsk

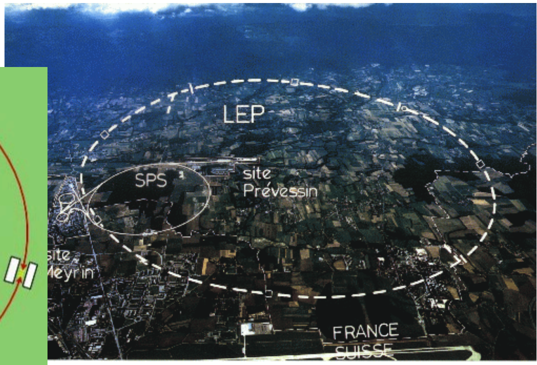
Particles: electron – electron
Collision energy: 160 MeV
Luminosity: 10^{28} 1/(cm^{2}s)
Rings size: two 1m x 1m

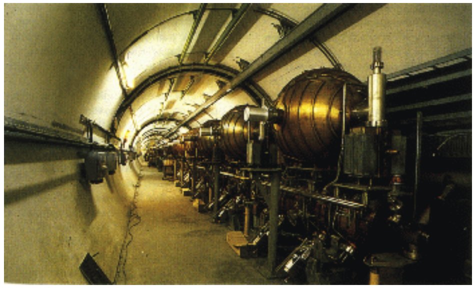


Large Electron–Positron Collider (LEP)
Operational: 1989 - 2000
Tunnel was used for LHC after LEP was decommissioned

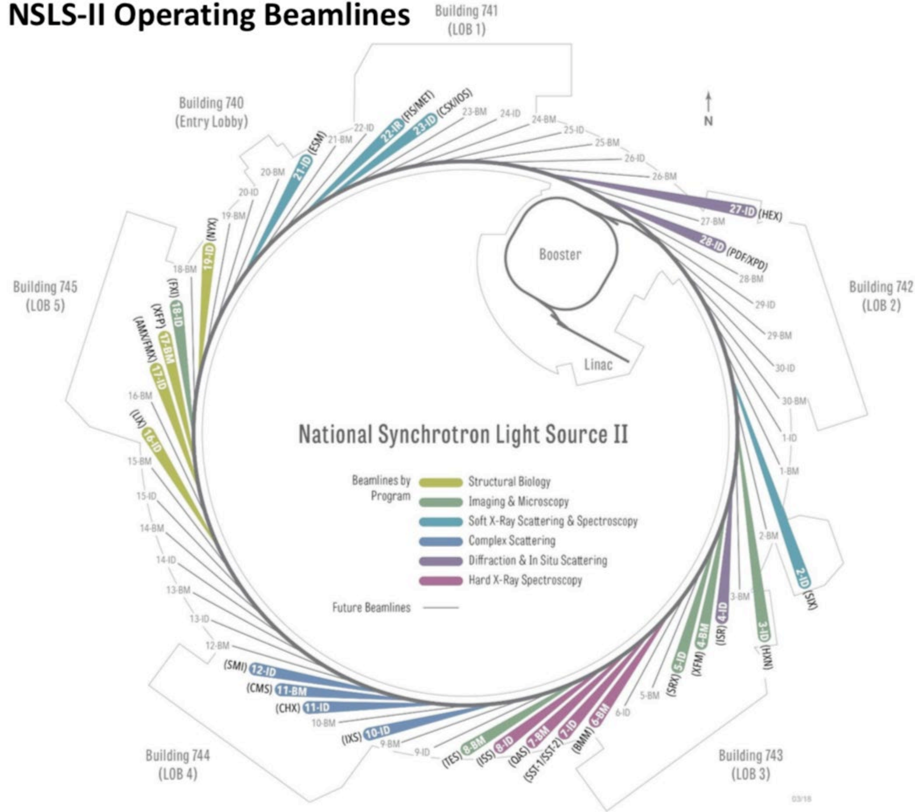
Particles: electron – positron
Collision energy: 100 GeV
Luminosity: 10^{32} 1/(cm^{2}s)
Circumference: 27 km







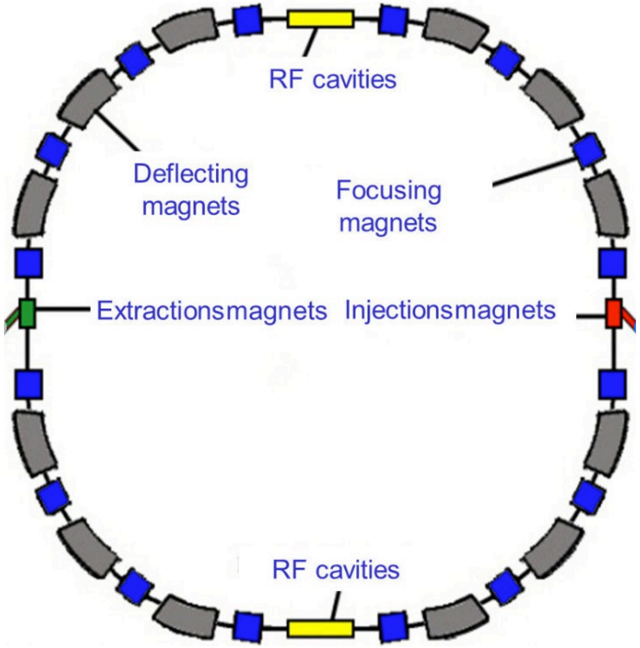
Introduction: Light Sources
Main Application of Modern Electron Rings



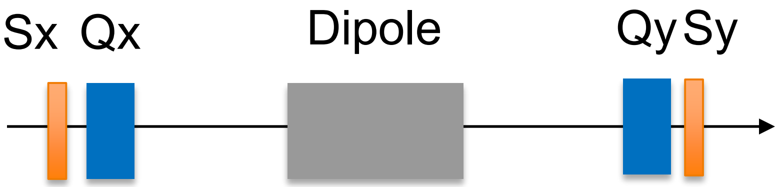
Particles: electrons
Energy: 3 GeV
Beam current: 0.5 A
Circumference: 792m
Number of bunches: 1056
Beam size (v/h): 3-13 μm / 30-150 μm
Experimental beamlines: 58

Simple Electron Ring Lattice and Typical Subcomponents

- Basic subcomponents of electron rings:
  - Bending magnets or electrostatic bends - dipoles
  - Focusing magnets – quads (can be incorporated into dipoles)
  - Multiple magnets to achieve specific beam dynamics characteristics
  - RF cavities to accelerate or compensate losses due to synchrotron losses and keep beam bunched
  - Injection/extraction systems

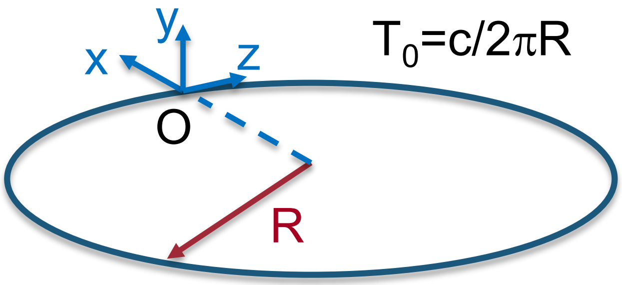


- Simplified lattice example
  - Bend
  - FODO doublet (Qx, Qy)
  - Sextupoles to compensate tune chromatism (Sx, Sy)



Accelerator Model Review

The single particle equations of motion have been derived previously.



$T_0 = c/2\pi R$

$$\{x, x', y, y', z, \delta; s\} \quad s \cong ct \qquad \delta = {}^{\Delta p}/_{p} \qquad \nu = \frac{1}{2\pi}\oint \frac{ds}{\beta}$$

$$x'' + \left(\frac{\nu_{x0}}{R}\right)^2 x = 0 \qquad k_{\beta x} = \frac{\nu_{x0}}{R} \qquad k_{\beta x}^2 = \frac{B'}{[B\rho]} + \frac{1}{\rho^2} \qquad B' = {}^{\partial B_y}/_{\partial x} \qquad \beta^2 = \frac{1}{k_\beta^2}$$

Transverse betatron motion

$$y'' + \left(\frac{\nu_{y0}}{R}\right)^2 y = 0 \qquad k_{\beta y} = \frac{\nu_{y0}}{R} \qquad k_{\beta y}^2 = -\frac{B'}{[B\rho]}$$

$$[B\rho] = \frac{p}{Q}$$ ‘Rigidity’

Transverse

$$z' = -\eta_{slip}\delta$$

‘Slippage’ $\eta_{slip} = \alpha - {}^{1}/_{\gamma^2}$ $\quad 0 < \alpha \approx {}^{1}/_{\nu_{x0}^2}\ (typically)$

Momentum compaction factor (lattice function)

$$\delta' = \begin{cases} 0 & , unbunched \\ \dfrac{1}{\eta_{slip}}\left(\dfrac{\nu_{s0}}{R}\right)^2 z & , bunched \end{cases}$$

$$\eta_{slip}\Big|_{tr} = 0 = \alpha - {}^{1}/_{\gamma_{tr}{}^2}$$

at transition

Longitudinal

Lattice Functions

- The lattice defines the environment in which the particles respond to perturbations

Dispersion

$$\binom{x_0}{p_0} \Longrightarrow \binom{x}{p} = \begin{pmatrix} D_x(p,s)\dfrac{\Delta p}{p_0} \\ p_0 + \Delta p \end{pmatrix} = \begin{pmatrix} D_x(p,s)\delta \\ p_0(1+\delta) \end{pmatrix}$$

$$\boxed{D_x'' + \left(k_{\beta x}^2 \frac{p_0}{p} - \frac{1}{\rho^2}\delta\right) D_x = \frac{1}{\rho}\frac{p_0}{p}}$$

η_x is also commonly used

Momentum Compaction (path length changes)

$$\frac{\Delta C}{C} = \alpha \frac{\Delta p}{p_0} = \alpha\delta \qquad \alpha = \left\langle \frac{D}{\rho} \right\rangle_{ring}$$

$C: specific\ orbit\ circumference$

Chromaticity, ξ (beware – definition varies!)

$$\Delta\nu = \xi(p)\frac{\Delta p}{p_0} = \xi\delta \qquad \nu = \frac{1}{2\pi}\oint \frac{ds}{\beta}$$

$$\boxed{Linac \quad \frac{\Delta k_\beta}{k_\beta} = \xi \frac{\Delta E}{E}}$$

Longitudinal Motion [1]

- Longitudinal motion is oscillatory and defined by the slippage factor

$$\eta_{slip} = \alpha - {}^{1}/_{\gamma^2}$$

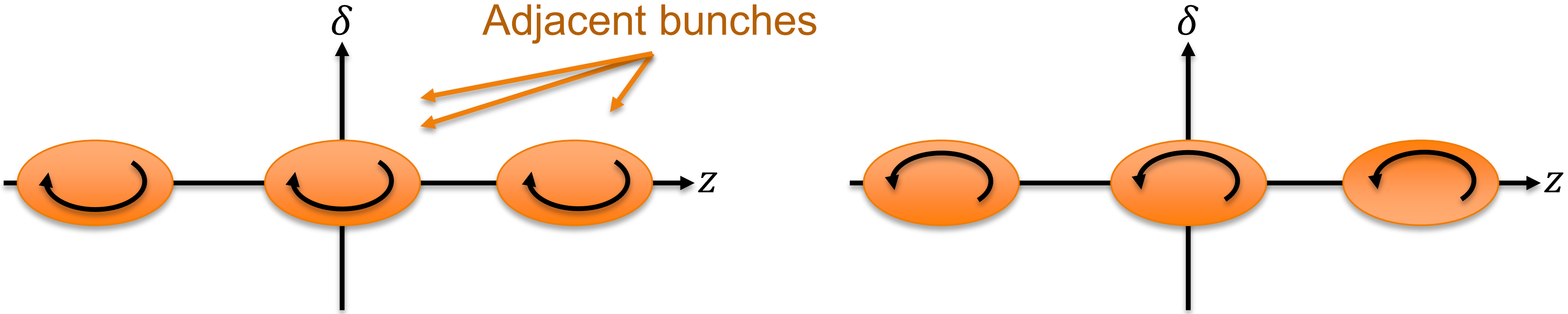
$$z' = -\eta_{slip}\delta$$

$$\delta' = \begin{cases} 0 & ,unbunched \\ \dfrac{1}{\eta_{slip}}\left(\dfrac{\nu_{s0}}{R}\right)^2 z & ,bunched \end{cases}$$

Longitudinal synchrotron motion

$$z'' + \left(\frac{\nu_{s0}}{R}\right)^2 = 0\,, \qquad bunched$$

$$\eta_{slip} = \alpha - {}^{1}/_{\gamma^2} = {}^{1}/_{\gamma_{tr}{}^2} - {}^{1}/_{\gamma^2}$$



Below transition energy, $\eta_{slip} < 0$

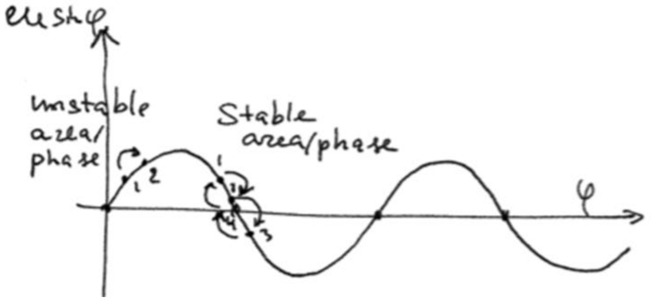
Above transition energy, $\eta_{slip} > 0$

Longitudinal Motion [2]

- Now with acceleration $V_{RF}(s) = V_{RF}\sin(\omega_{RF}t)$

Synchronous particle: energy gain = energy losses / revolution

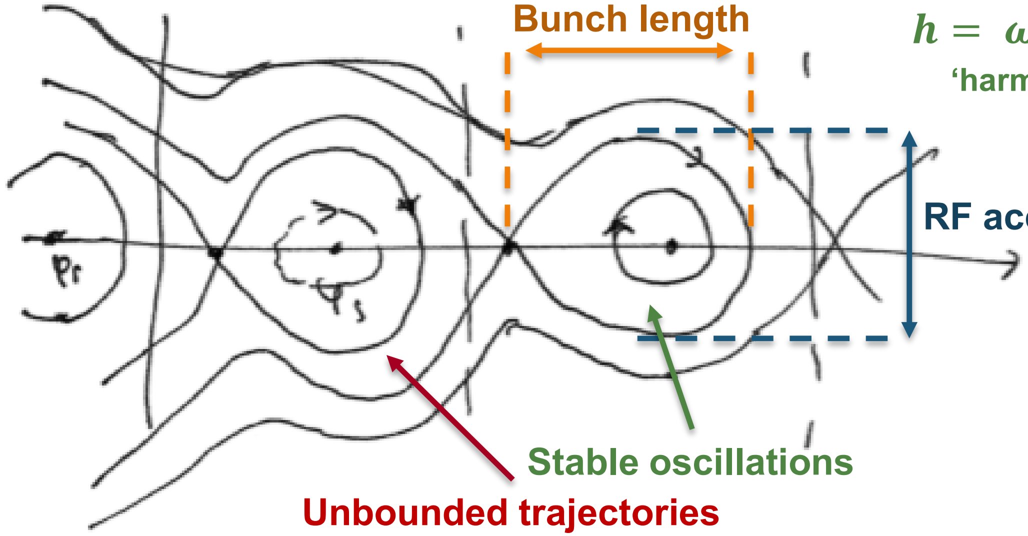
$$eV_{RF}\sin(\phi_s) = U_0$$ ← primarily synchrotron radiation losses



$\varphi_1 < \varphi_2$

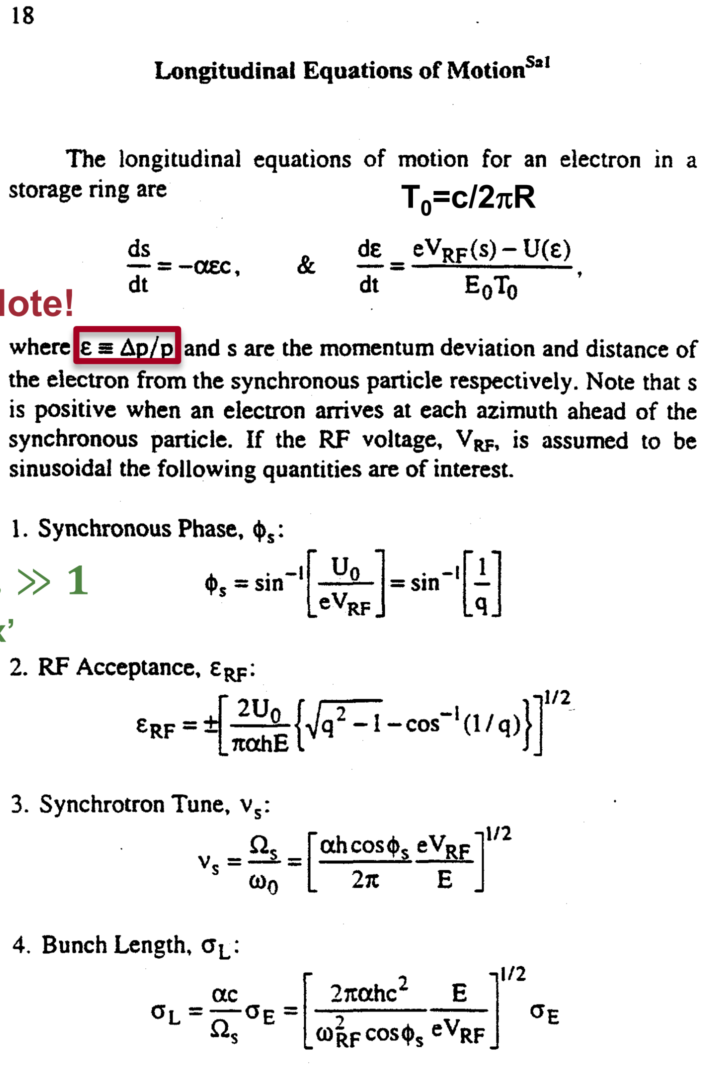
means

particle 1 arrives earlier than particle 2



$h = \omega_{RF}/\omega_{rev} \gg 1$

'harmonic index'



18

Longitudinal Equations of Motion[Sa1]

The longitudinal equations of motion for an electron in a storage ring are

$T_0=c/2\pi R$

$$\frac{ds}{dt} = -\alpha\varepsilon c, \quad \& \quad \frac{d\varepsilon}{dt} = \frac{eV_{RF}(s) - U(\varepsilon)}{E_0T_0},$$

Note!

where $\boxed{\varepsilon \equiv \Delta p/p}$ and s are the momentum deviation and distance of the electron from the synchronous particle respectively. Note that s is positive when an electron arrives at each azimuth ahead of the synchronous particle. If the RF voltage, V_{RF}, is assumed to be sinusoidal the following quantities are of interest.

1. Synchronous Phase, ϕ_s:

$$\phi_s = \sin^{-1}\left[\frac{U_0}{eV_{RF}}\right] = \sin^{-1}\left[\frac{1}{q}\right]$$

2. RF Acceptance, ε_{RF}:

$$\varepsilon_{RF} = \pm\left[\frac{2U_0}{\pi\alpha hE}\left\{\sqrt{q^2-1} - \cos^{-1}(1/q)\right\}\right]^{1/2}$$

3. Synchrotron Tune, ν_s:

$$\nu_s = \frac{\Omega_s}{\omega_0} = \left[\frac{\alpha h\cos\phi_s}{2\pi}\frac{eV_{RF}}{E}\right]^{1/2}$$

4. Bunch Length, σ_L:

$$\sigma_L = \frac{\alpha c}{\Omega_s}\sigma_E = \left[\frac{2\pi\alpha hc^2}{\omega_{RF}^2\cos\phi_s}\frac{E}{eV_{RF}}\right]^{1/2}\sigma_E$$

J. Murphy, ed., Synchrotron Light Source Data Booklet, v.4, 1996

Solutions to Hill's Equation

- The linear motion has known solutions

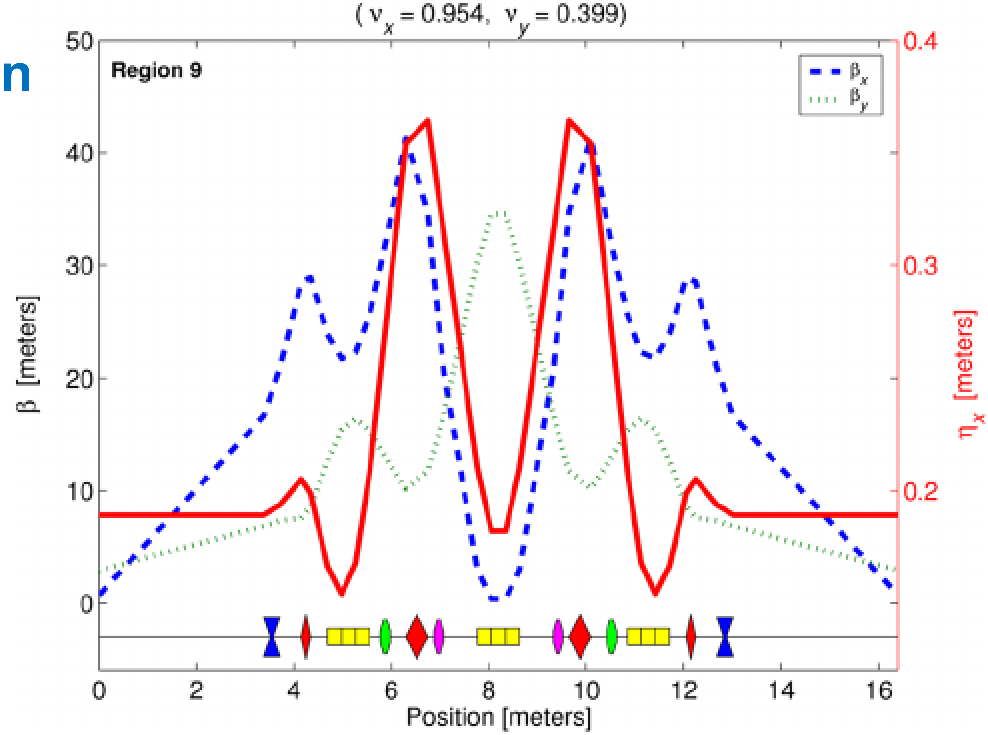
$$x_\beta'' + \left(\frac{\nu_{x0}}{R}\right)^2 x_\beta = 0$$ **Hill's Equation**

$$x_\beta'' + k_{\beta x}^2 x_\beta = 0 \implies$$

$$x_\beta(s)$$

$$= \sqrt{\beta_x(s)} \cos\left[\int \frac{ds}{\beta_x} + \vartheta_x\right]$$

$$= \hat{x}\sqrt{\frac{\beta_x(s)}{\beta_0}} \cos(\psi_x(s))$$



D. Robin, et al, PhysRevSTAB.11.024002.

Advanced Light Source (1 of 12 Sectors)

Natural Chromaticity

- The presence of energy spread in the beam leads to variations in the betatron tune.
- Any lattice has a 'natural' chromaticity $\xi_{natural} = -\frac{1}{4\pi}\oint k_{\beta}^{2}\beta ds$

$$k_{\beta x} = \frac{\nu_{x0}}{R} \qquad k_{\beta x}^{2} = \frac{B'}{[B\rho]} + \frac{1}{\rho^{2}}$$

- Uncorrected, this natural chromaticity results in strong variations in betratron functions with energy deviations.
- Negative chromaticity is also to be avoided so that head-tail instabilities and coupled-bunch oscillations may be suppressed.
- The addition of a sextupole magnet (length, *l*) is commonly used to correct the natural chromaticity. But this introduces *nonlinearity* into the ring dynamics.

$$\Delta\xi_{sext} = \pm\frac{1}{4\pi}D\beta\frac{B''l}{[B\rho]}$$

(+) for bend plane (eg. horizontal)

(-) for out-of-plane (eg. vertical)

Introduction to Lattice Perturbations

- Non-ideal elements and symmetry-breaking insertions provide localized sources of perturbations

$$y'' + \left(\frac{\nu_{y0}}{R}\right)^2 y = \underbrace{-Ky}_{\text{example perturbation}} \implies y'' + \underbrace{\left[\left(\frac{\nu_{y0}}{R}\right)^2 - K\right]}_{= \left(\frac{\nu_y}{R}\right)^2} y = 0$$

$$\nu_y^2 = \nu_{y0}^2 - KR^2$$

- Small perturbation –> orbit distortion

$$\Delta\nu_y = \nu_y - \nu_{y0} \approx -\frac{KR^2}{2\nu_{y0}} \qquad y_\beta(s) = \hat{y}\sqrt{\frac{\beta_y(s) + \Delta\beta_y}{\beta_0}}\cos\left(\psi_y(s)\right)$$

- Large perturbation –> secular growth, nonlinear island formation

$$\nu_y^2 = \nu_{y0}^2 - KR^2 < 0$$

Linear coupling between planes

- Coupling between planes from skew and solenoidal components, misalignments, etc.

$$x = x_0 + x_\beta + x_D = x_0 + \hat{x}\sqrt{\frac{\beta_x}{\beta_0}}\cos(\psi_x) + D_x\,\delta$$

$$y = y_0 + \; y_\beta + y_D = y_0 + \hat{y}\sqrt{\frac{\beta_y}{\beta_0}}\cos(\psi_y) + D_y\delta$$

$$x'' + k_{\beta x}^2 x = Sy + Ry' + \frac{1}{2}R'y + \cdots \qquad S(s) = \frac{B'_{skew}}{[B\rho]} = \frac{\partial B_x/\partial x}{[B\rho]}$$

$$y'' + k_{\beta y}^2 y = Sx - Rx' - \frac{1}{2}R'x + \cdots \qquad R = \frac{B_{sol}}{[B\rho]}$$

- Analysis of motion (similar to development of Courant-Snyder parameters) finds at lowest order of perturbation

$\hat{x}^2 + \hat{y}^2 = constant \;\; \nu_x - \nu_y = Integer$ Difference resonance, bounded

$\hat{x}^2 - \hat{y}^2 = constant \;\; \nu_x + \nu_y = Integer$ Sum resonance, **unbounded**

Tune Diagram with Resonances

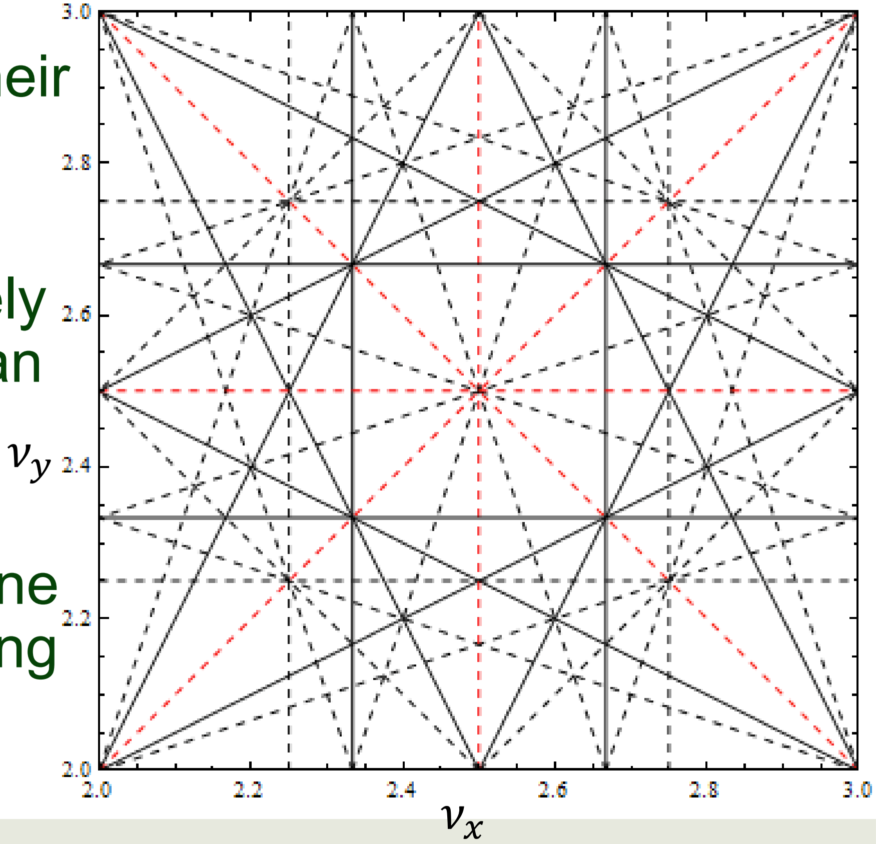
- In general, the resonances happen when tunes satisfy equation

$$k\nu_x + l\nu_y = m$$

$$k, l, m \ - integers$$

- The strength of the resonances and their destructive effects reduce with the resonance order (m)
- Resonances higher than 4th order rarely cause instantaneous beam loss but can cause emittance increase and beam quality reduction.
- Resonance harmonics equal to machine periodicity (q) can be particularly strong (ie. excited at every lattice cell)

Tune Diagram between 2 and 3 for $k, l \ \leq 4$



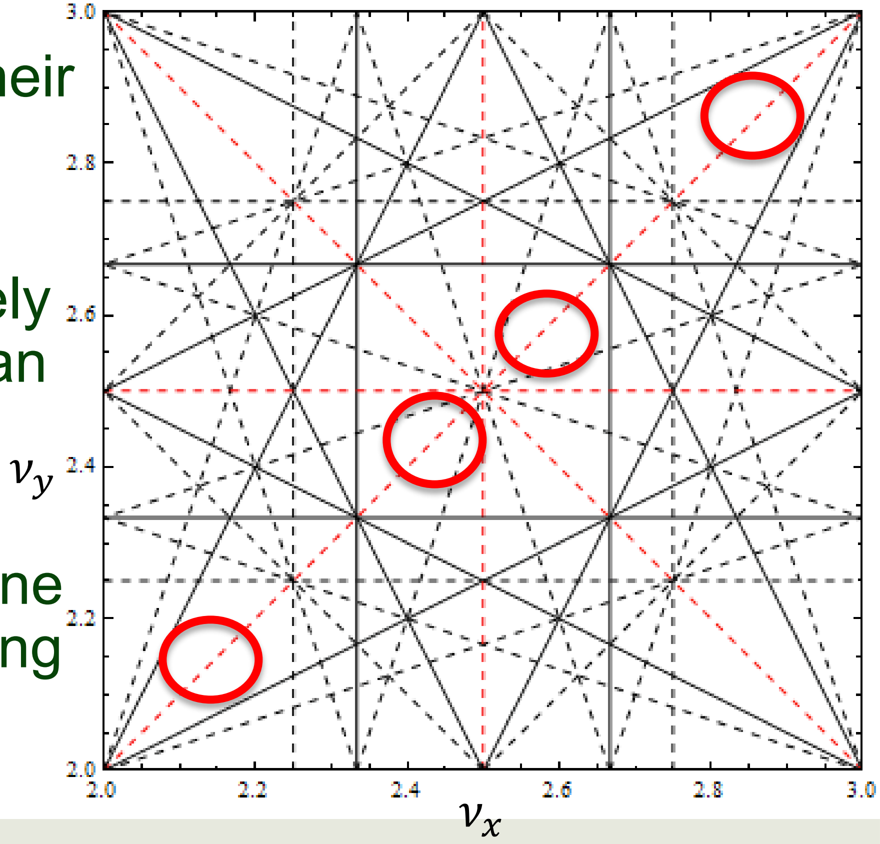
Tune Diagram with Resonances

- In general, the resonances happen when tunes satisfy equation

$$k\nu_x + l\nu_y = m$$
$$k, l, m \; - integers$$

- The strength of the resonances and their destructive effects reduce with the resonance order (m)
- Resonances higher than 4th order rarely cause instantaneous beam loss but can cause emittance increase and beam quality reduction.
- Resonance harmonics equal to machine periodicity (q) can be particularly strong (ie. excited at every lattice cell)

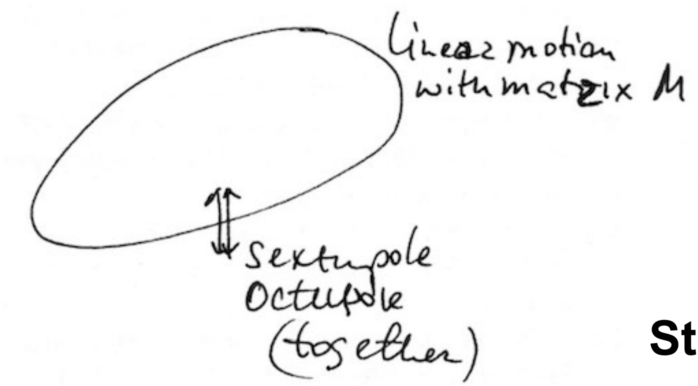
Red circles show approximate area typically used by electron ring synchrotrons for operations.



Non-Linear Dynamics and Its Treatment

- Nonlinear elements can severely affect beam dynamics in the rings
  - Cause fast beam losses and beam quality degradation
  - Limit beam lifetime in an accelerator
  - Limit suitable selection of betatron tunes
- Accurate treatment of nonlinear motion still is not possible. There is no mathematical apparatus that would allow us to do that in a general case (except some specific cases)
- Iterative perturbation analysis and averaging are used and produce good results. However, this treatment is beyond the scope of the course (although it is not too complicated and relies on analysis of corresponding Hamiltonian Functions. It is just time consuming.)
- We study a simple model numerically to get a qualitative picture

Numerical Model and Motion Far From Resonances



Step 1 – one turn transformation, linear optics

$$\begin{bmatrix} x \\ x' \end{bmatrix}_2 = \begin{bmatrix} \cos(2\pi\nu_x) & \sin(2\pi\nu_x) \\ -\sin(2\pi\nu_x) & \cos(2\pi\nu_x) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_1$$

Step 2 – thin sextupole and octupole transformations

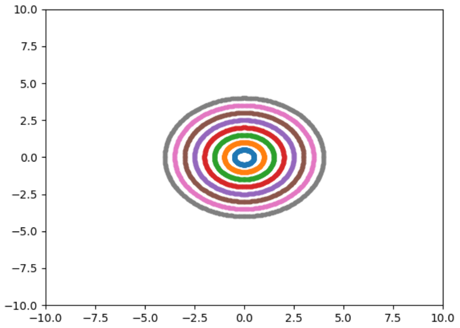
$$\begin{bmatrix} x \\ x' \end{bmatrix}_3 = \begin{bmatrix} 0 \\ Sx_2^2 + Ox_2^3 \end{bmatrix}$$

Sextupole term

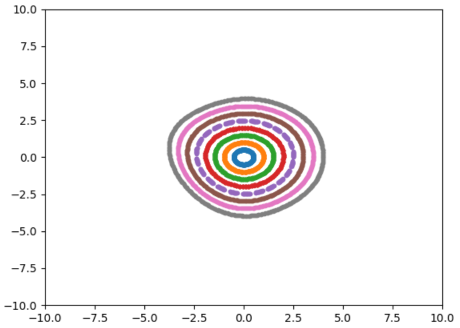
Octupole term

$\nu = 0.171$ - far from resonances, motion with nonlinearities is perturbed but not dramatically. Linear motion shows no perturbations (ellipse).

Linear



With Nonlinearities



S= 0.05, O= -0.01

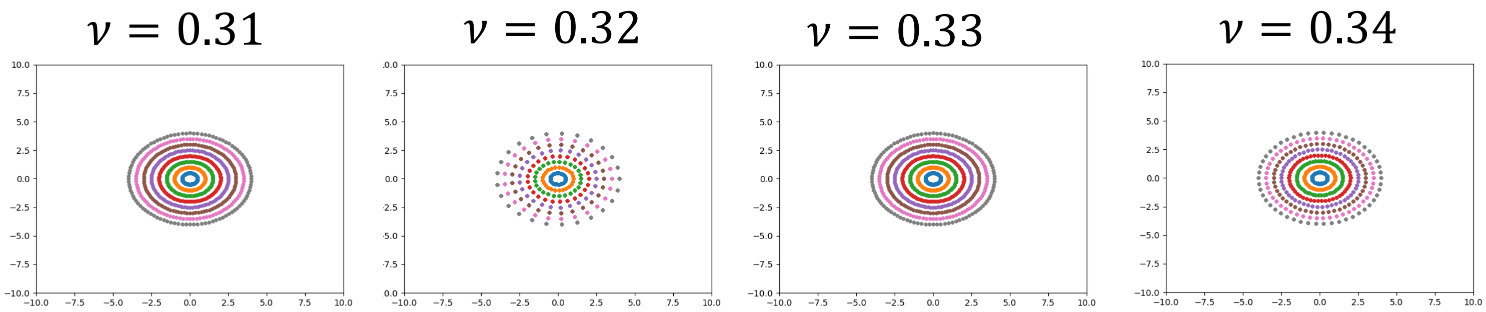
$$\frac{\partial \nu}{\partial A^2} > 0 \quad \text{for O < 0}$$

Tune shift is positive for large amplitudes

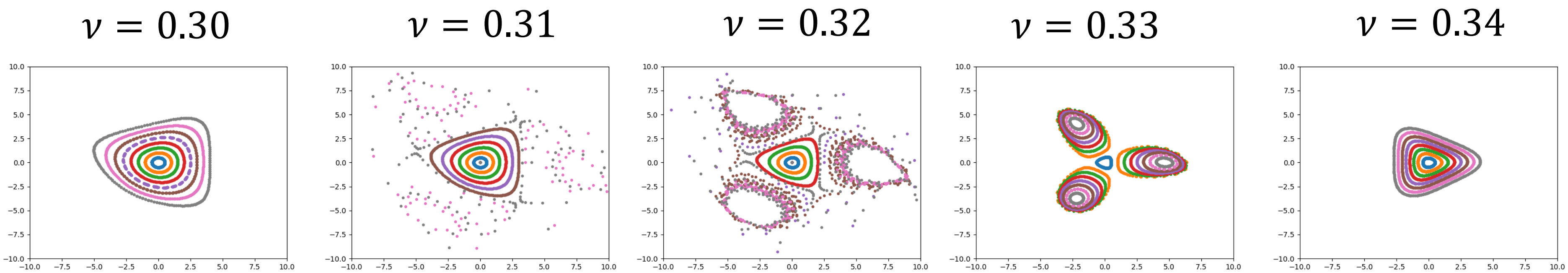
$\nu = q/3$ Resonance
(in horizontal x-x' phase space plane)

Linear motion, sext = 0, oct = 0 – no phase space perturbation

$3\nu_x = q$



Non-linear motion, Sext = 0.05, Oct = -0.01 – strong perturbation of phase space. Particles become unstable (Amplitude → ∞), causing losses in a few turns

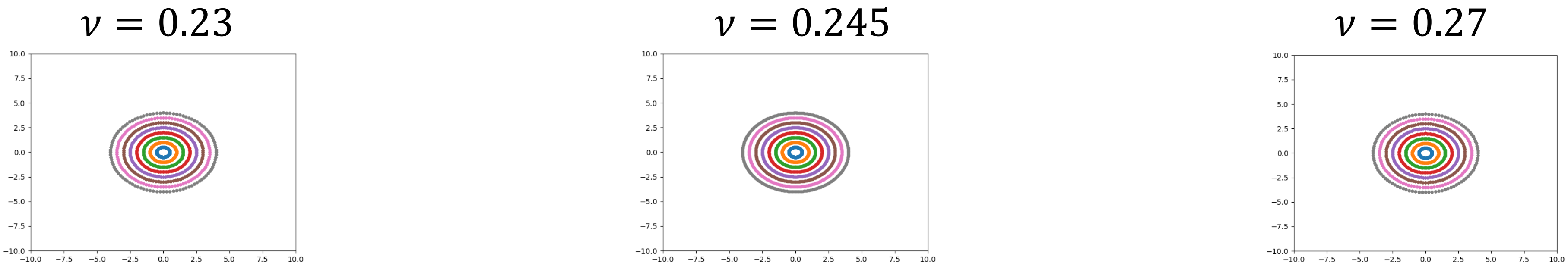


Particles with larger amplitudes get have a higher frequency, see previous slide

$\nu = q/4$ Resonance

(in horizontal x-x’ phase space plane)

Linear motion, sext = 0, oct = 0 – no phase space perturbation

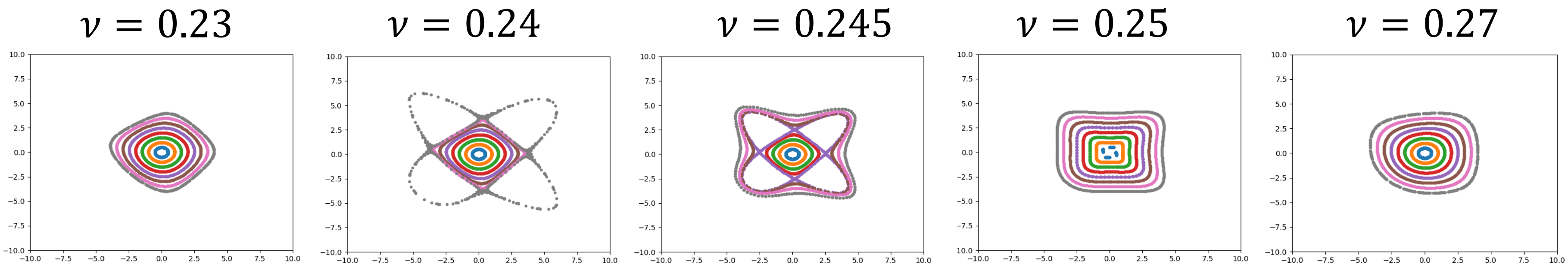


$\nu = 0.23$

$\nu = 0.245$

$\nu = 0.27$

Non-linear motion, sext = 0.05, oct = -0.01 – strong perturbation of phase space



$\nu = 0.23$

$\nu = 0.24$

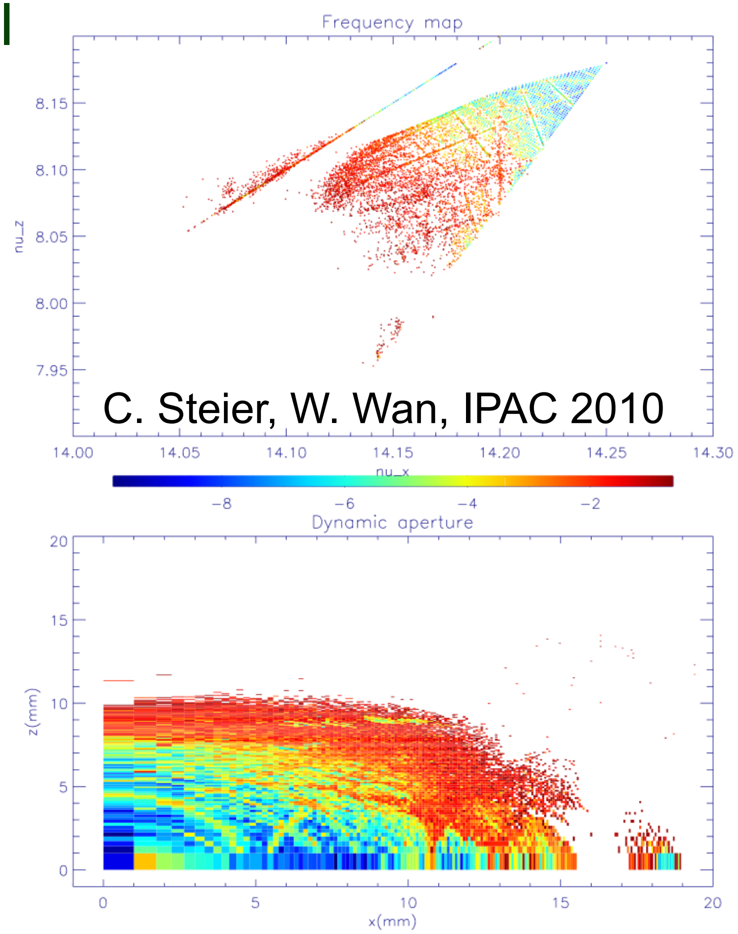
$\nu = 0.245$

$\nu = 0.25$

$\nu = 0.27$

Frequency Map Analysis

- Frequency map analysis is a very powerful tool to understand and improve the nonlinear dynamic behavior in particle accelerators.
- Frequency map analysis is used to compare the performance of different lattices and to carry out an automated lattice optimization.
- Experimentally, ‘pinger’ magnets are used to excite motion and explore areas of the nonlinear dynamic aperture. Turn-by-turn motion is measured with BPMs.
- See http://www.cpt.univ-mrs.fr/~hscopp04/Abstracts/Laskar.pdf

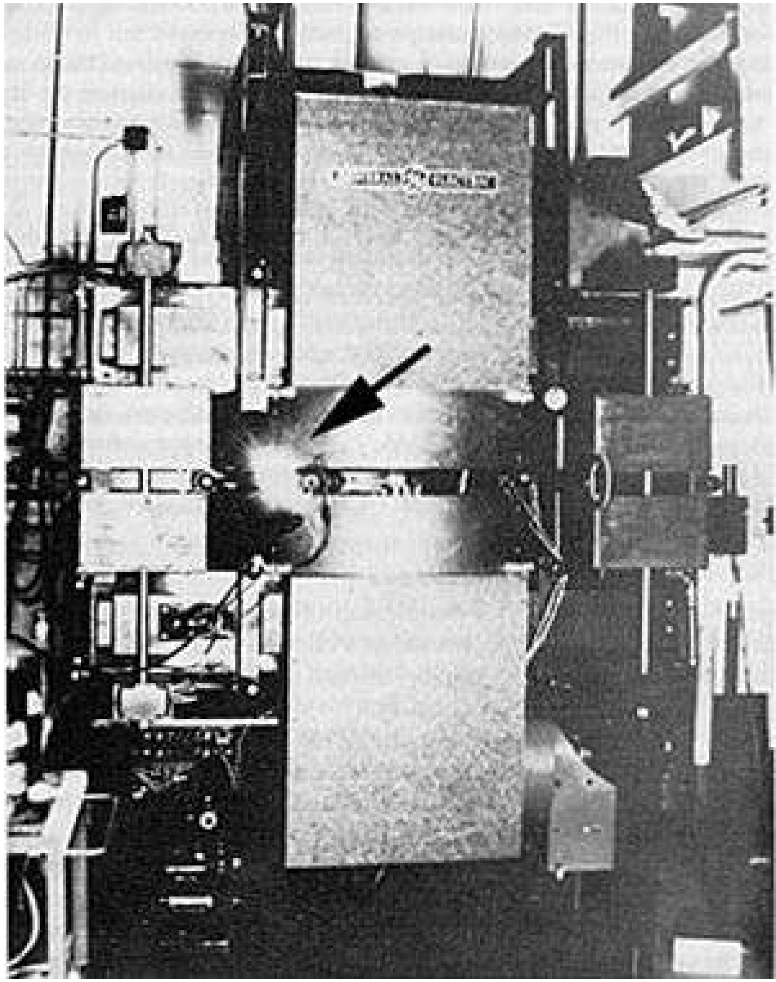


C. Steier, W. Wan, IPAC 2010

Figure 1: Simulated Frequency Map for the ALS lattice with errors in configuration and frequency space.

Synchrotron Radiation [1]

- Synchrotron radiation is a by-product of transverse acceleration of charged particles.
- Predicted by Ivanenko and Pomeranchuk in 1943.
- Observed in 1947 in General Electric electron synchtrotron.
- Originally considered a nuisance as it provides a channel to drain energy from the stored beam – with a strong dependence on beam energy.
- Nowadays it provides the basis of incredibly useful facilities for scientific discovery.



Synchrotron Radiation [2]

- Radiation is emitted by relativistic charged particles due to acceleration in a magnetic field.
- Radiation is quantum in nature, but the high intensity of the field leads to classical analysis.
- Radiation is emitted over a broad spectrum of low photon energies and falls off exponentially above the critical energy

$$\epsilon_{crit} = \hbar\omega_{crit} = \frac{3\hbar c\gamma^3}{2\rho} \Longrightarrow \epsilon_{crit}[keV] = 0.665\, B[T]E^2[GeV]$$

- The total power radiated is given by

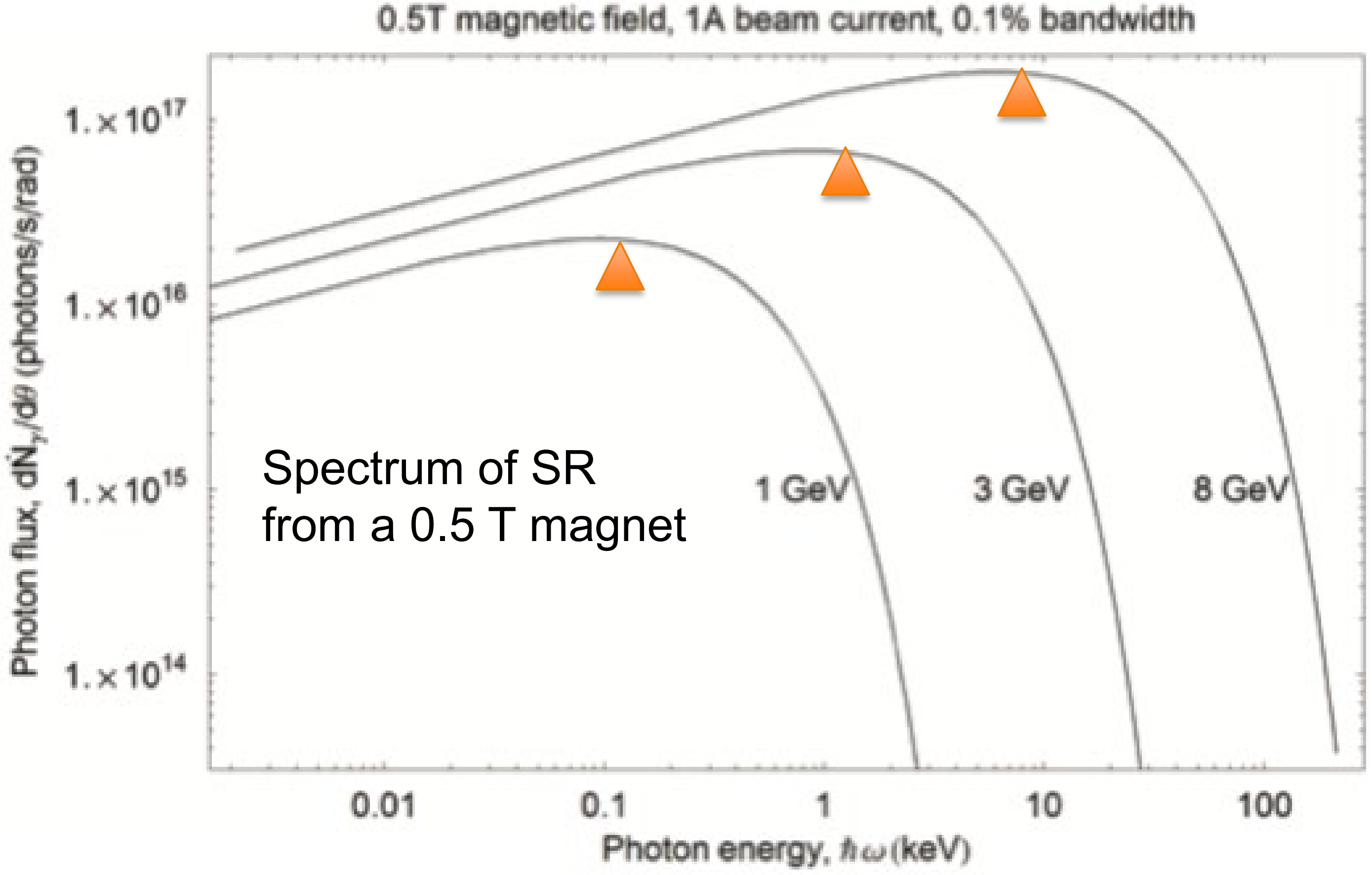
$$P_{total} = \frac{4\pi r_e mc^2}{3e}\frac{\gamma^4}{\rho}I \;\Longrightarrow\; P_{total}[kW] = U_0[keV]I[A] = \left[\frac{88.5\, E^4[GeV]}{\rho[m]}\right]I[A]$$

- This power must be replenished by the synchrotron's RF system

Spectrum of Synchrotron Radiation

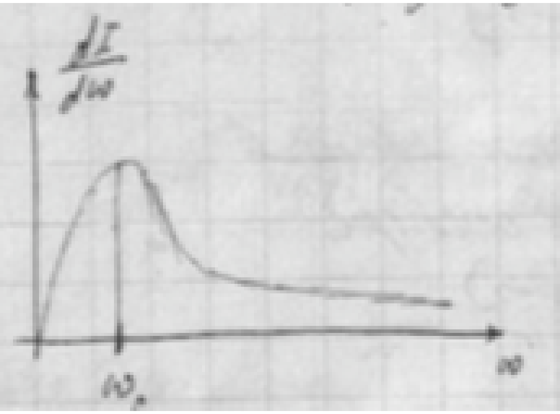
$$\epsilon_{crit} = \hbar\omega_{crit} = \frac{3\hbar c\gamma^3}{2\rho}$$

Characteristic energy of SR spectrum



Spectrum of SR
from a 0.5 T magnet

Quantum Nature of Synchrotron Radiation

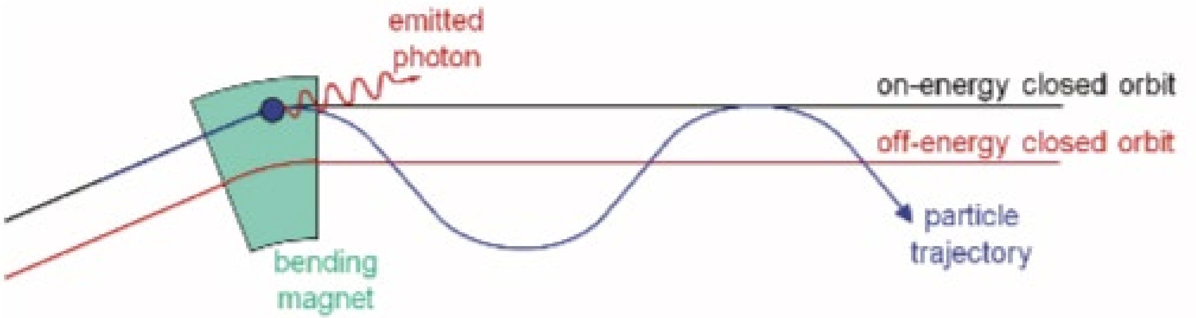


Number of photons emitted per turn $N \approx \alpha\gamma = \frac{\gamma}{137}$

α- is the fine-structure constant

Statistical emission of a quantum appears as a change in an equilibrium orbit by recoil, causing oscillations around that new orbit - increases betatron oscillations.

Multiple emissions behave like Brownian motion causing diffusion and increase of emittance.



Quantum oscillations ultimately limit the equilibrium emittance.

The equilibrium emittance is defined by the damping rate and by the growth rate caused by random emissions of light quanta.

Quantum Statistics of Synchrotron Emission

Parameter	Value
Mean photon energy, $\langle\epsilon\rangle$	$\frac{8}{15\sqrt{3}}\epsilon_{crit}$
RMS photon energy, $\langle\epsilon^2\rangle$	$\frac{11}{27}\epsilon_{crit}^2$
Total photon flux, $\dot{N}_{ph}$	$\frac{15\sqrt{3}}{8}\frac{P_\gamma}{\epsilon_{crit}}$
Product, $\dot{N}_{ph}\langle\epsilon^2\rangle$	$\frac{55}{24\sqrt{3}}P_\gamma\epsilon_{crit} = \frac{55}{24\sqrt{3}}\hbar c^2 r_e m c^2 \frac{\gamma^7}{\lvert\rho^3\rvert}$

Quantum excitation over path length, L

$$\Delta\sigma_E^2\Big|_{quant} = \frac{55(\hbar c)^2}{48\sqrt{3}}\gamma^7\int_0^L\left(\frac{1}{\lvert\rho_x^3\rvert}+\frac{1}{\lvert\rho_y^3\rvert}\right)ds$$

Emittance increase over path length, L

$$\Delta\varepsilon_u\Big|_{quant} = \frac{55 r_e\hbar c}{48\sqrt{3}mc^2}\gamma^5\int_0^L\frac{\mathcal{H}_u}{\lvert\rho_u^3\rvert}$$

Radiation Damping

- Emission of synchrotron radiation reduces the electron energy.
- An electron radiates at the average rate U_0/T_0 where $T_0=c/2\pi R$ is the average revolution time.
- Electrons on different betatron oscillations, but with the same energy, will lose the same amount of energy (when averaged, in linear approximation)
- Electrons with different energies, will radiate different amounts
- Electrons emit photons within an angle $1/\gamma$ of the forward motion
  - Longitudinal momentum is replaced by RF acceleration
  - Transverse momentum is damped

M. Sands, SLAC-121 (1970)

Damping of Synchrotron Oscillations

$$E_{i+1} = E_i + eU\cdot\sin\varphi - W =$$

$$= E_i + eU\cdot\sin\varphi - \left(W_0 + \frac{dW}{dE}\varepsilon_i\right)$$

$$\varepsilon_i = E_i - E_s$$

Energy transformation after 1 turn for electron with energy deviated from the synchronous energy

$$E_s = E_s + eU\cdot\sin\varphi_s - W_0$$

Energy and phase of synchronous particle

$$\Rightarrow \quad \frac{d\varepsilon}{dt} = \frac{eU}{T}(\sin\varphi - \sin\varphi_s) - \frac{dW}{TdE}\varepsilon$$

$$\Rightarrow \quad \frac{d^2\varepsilon}{dt^2} + \frac{1}{T}\frac{dW}{dE}\cdot\frac{d\varepsilon}{dt} + \omega_s^2\cdot\varepsilon = 0$$

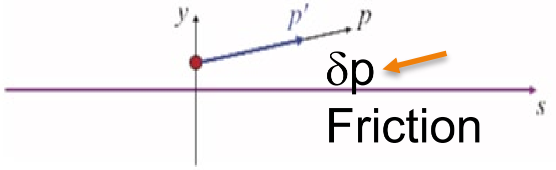
For small oscillations

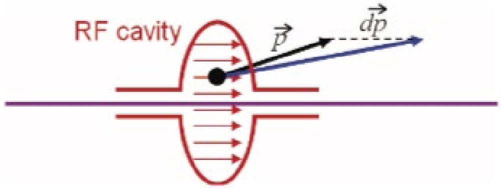
$$\zeta_s = \frac{1}{\tau_s} = \frac{1}{T}\frac{dW}{dE} = \frac{W_0}{2ET}(2+D)$$

$$D = \frac{\int \frac{\eta}{r}\left(\frac{1}{r^2} + \frac{2eB'}{pc}\right)}{\int 1/r^2\, ds}$$

r - bending ~~radii~~ radius

Damping of Vertical Oscillations





$y'' + K y = 0$ – derivative $\frac{d}{ds}$

$\ddot{y} + \omega_{\beta y}^2 y = 0$ – derivative $\frac{d}{dt}$

Energy loss because of SR friction $F = -\frac{P}{v} \cdot \frac{\vec{v}}{v}$

$$\ddot{y} + \frac{P_{SR}}{\gamma m c^2} \dot{y} + \omega_{\beta y}^2 \cdot y = 0$$

$$J_y = \frac{1}{\tau_y} = \frac{P_{SR}}{2E_0} = \frac{W_0}{2E_0 T_0}$$

W_0 – energy loss per turn

$$J_y = \frac{2\pi}{3} \frac{r_{0e}}{r} \frac{\gamma^3}{T_0}$$

r_{0e} – classical electron radius.

$$T_\beta \ll T_0 \ll T_{Synch} \ll T_{damp.} \ll T_{LT}$$

Betatron osc. period

Rev time

synchrotron oscillations

SR damping

lifetime

Beam Lifetime

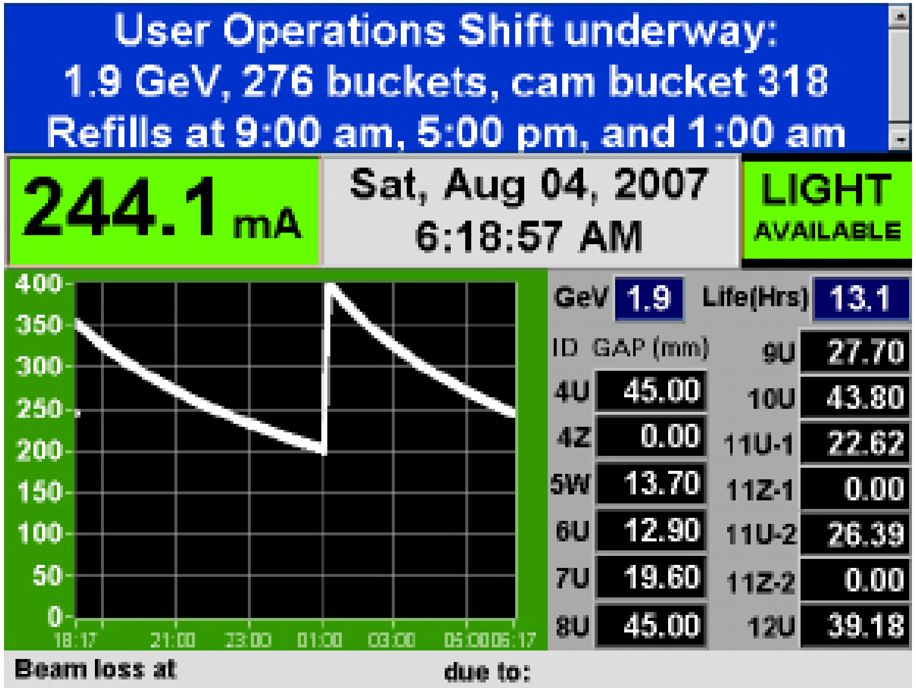
	$\boldsymbol{\tau_{total}^{-1}}$	$\boldsymbol{= \tau_{scat}^{-1} + \tau_{brem}^{-1} + \tau_{Tous}^{-1} + \tau_{quant}^{-1}}$
Gas scattering	τ_{scat}^{-1}	$\frac{4r_e^2 Z^2 \pi\rho c}{2\gamma^2}\left[\frac{\langle\beta_x\rangle\beta_{x,max}}{a^2} + \frac{\langle\beta_y\rangle\beta_{y,max}}{b^2}\right]$
Bremsstrahlung on nuclei	τ_{brem}^{-1}	$\frac{16r_e^2 Z^2 \rho c}{411} ln\left[\frac{183}{Z^{1/3}}\right]\left[-ln\varepsilon_{RF} - \frac{5}{8}\right]$
Touschek half-life	τ_{Tous}^{-1}	$\frac{\sqrt{\pi} r_e^2 cNC(\zeta)}{\sigma_x' \gamma^3 \varepsilon_{acc}^2 V}, V = 8\pi^{3/2}\sigma_x\sigma_y\sigma_z, \zeta = (\varepsilon_{acc}/\gamma\, \sigma_x')^2$
Quantum	τ_{quant}	$\frac{\tau_s}{2}\frac{e^{\xi}}{\xi}, \quad \xi = {\varepsilon_{rf}^2}\big/{2\sigma_E^2}$

J. Murphy, ed., Synchrotron Light Source Data Booklet, v.4, 1996

Top Off Mode

Continuous replacement of lost beam

Advanced Light Source



User Operations Shift underway:
1.9 GeV, 276 buckets, cam bucket 318
Refills at 9:00 am, 5:00 pm, and 1:00 am

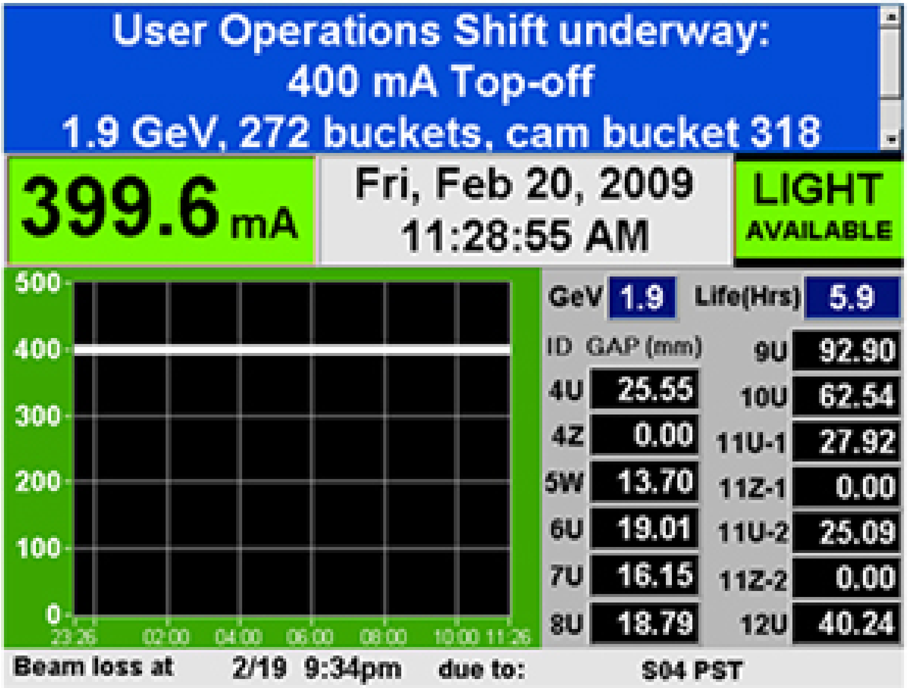
244.1 mA

Sat, Aug 04, 2007
6:18:57 AM

LIGHT AVAILABLE

GeV	1.9	Life(Hrs)	13.1
ID GAP (mm)		9U	27.70
4U	45.00	10U	43.80
4Z	0.00	11U-1	22.62
5W	13.70	11Z-1	0.00
6U	12.90	11U-2	26.39
7U	19.60	11Z-2	0.00
8U	45.00	12U	39.18

Beam loss at due to:



User Operations Shift underway:
400 mA Top-off
1.9 GeV, 272 buckets, cam bucket 318

399.6 mA

Fri, Feb 20, 2009
11:28:55 AM

LIGHT AVAILABLE

GeV	1.9	Life(Hrs)	5.9
ID GAP (mm)		9U	92.90
4U	25.55	10U	62.54
4Z	0.00	11U-1	27.92
5W	13.70	11Z-1	0.00
6U	19.01	11U-2	25.09
7U	16.15	11Z-2	0.00
8U	18.79	12U	40.24

Beam loss at 2/19 9:34pm due to: S04 PST

Acknowledgments

- Some material (mostly pictures) were “borrowed” from the USPAS 2013 school course “Design of Electron Storage and Damping Rings” by Andy Wolski and David Newton, USPAS, Fort Collins, Colorado, 2013
- SPRING 8 informational video available on YouTube

Appendix 1

Transverse and Longitudinal Motion in Electron Rings

Equations of Motion and Hill Equation

(x, y) - small deviations from the reference particle

s is the independent variable instead of t (s=v*t)

$$\frac{d^2x}{ds^2} + \frac{1-n}{r^2}x = \frac{1}{r}\left(\frac{\Delta p}{p} - \frac{\Delta B}{B}\right)$$
$$\frac{d^2y}{ds^2} + \frac{n}{r^2}y = 0$$

- Dipole magnet with gradient focusing

n is the field index $n = -\frac{r_0}{B}\left(\frac{dB}{dr}\right)_0$

$$\frac{d^2x}{ds^2} + \frac{eB'x}{cp} = 0$$
$$\frac{d^2y}{ds^2} - \frac{eB'y}{cp} = 0$$

- Quadrupole

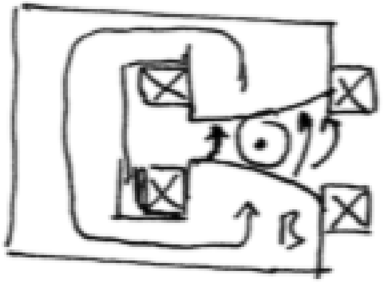
$$\frac{d^2x}{ds^2} = 0$$
$$\frac{d^2y}{ds^2} = 0$$

- Drift

$$x'' + k(s)x = 0$$
$$k(s) = k(s+C)$$

- General Hill equation with periodic focusing

Example: Weak Focusing Azimuthally Symmetric Field with Gradient



$$\frac{1-n}{r^2} - \text{constant}$$

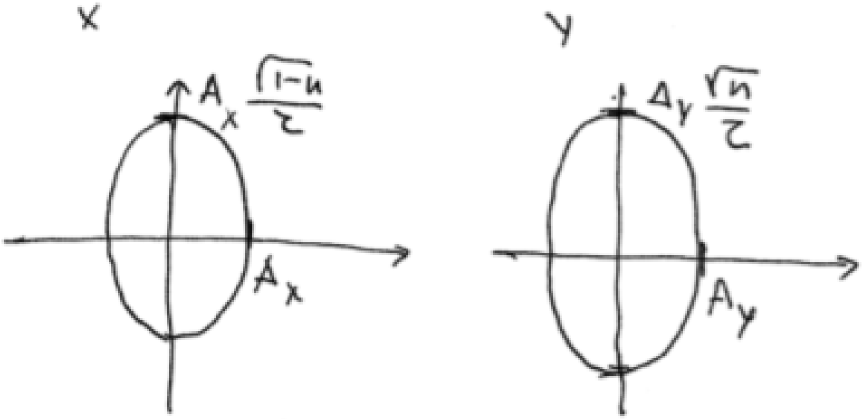
$$\frac{n}{r^2} - \text{constant}$$

$$n = -\frac{r}{B}\frac{dB}{dr}$$

$$x = A_x \cos\left(\frac{\sqrt{1-n}}{r} \cdot s + \varphi_x\right) \qquad x' = -A_x \sin\left(\frac{\sqrt{1-n}}{r} s + \varphi_x\right)\frac{\sqrt{1-n}}{r}$$

$$y = A_y \cdot \cos\left(\frac{\sqrt{n}}{r} s + \varphi_y\right) \qquad y' = -A_y \sin\left(\frac{\sqrt{n}}{r} s + \varphi_y\right) \cdot \frac{\sqrt{n}}{r}$$

Solution
easily obtainable



$$S_x = \pi A_x^2 \cdot \frac{\sqrt{1-n}}{r} \qquad S_y = \pi A_y^2 \cdot \frac{\sqrt{n}}{r}$$

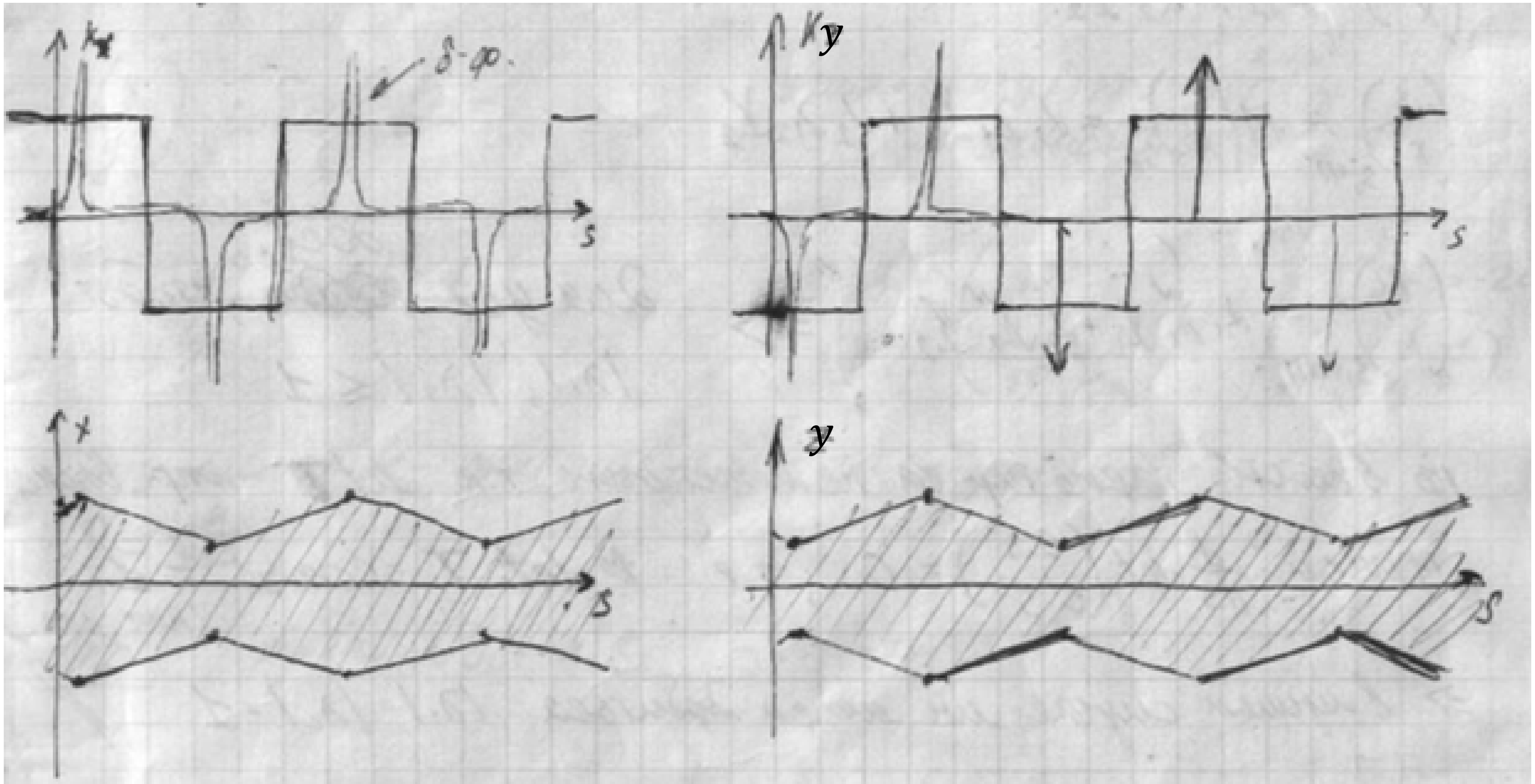
Current is phase space density times area

1. Increase density
2. Increase aperture
3. Increase focusing

Increasing focusing in both planes is Impossible. Need other focusing Mechanism (strong focusing)

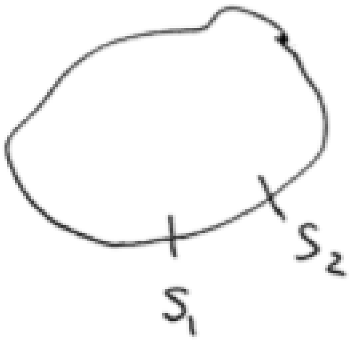
Strong Focusing

Strong focusing can be achieved by introducing variable focusing as function of s. However, stability and properties of such motion needs to be investigated.



Linear Betatron Motion

Linear motion can be described by vectors and matrices



Cosine like solution $C(s_1)=1$, $C'(s_1)=0$

Sine like solution $S(s_1)=0$, $S'(s_1)=1$.

$$\begin{bmatrix} x \\ x' \end{bmatrix}_2 = \underbrace{\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix}}_{T(s_1,\, s_2)} \begin{bmatrix} x \\ x' \end{bmatrix}_1$$

$$\left(\begin{bmatrix} x \\ x' \end{bmatrix}_1 = C(s_1)x_1 + S(s_1)x_1' \right)$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1'$$

$T(s, s+C) = M$ – one turn matrix

Stability of Betatron Motion [1]

$T(s, s+C) = M$ – one turn matrix

Y_1 and Y_2 - Eigen vectors of M (basis) with eigen values λ_1 and λ_2.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \alpha_1 Y_1 + \alpha_2 Y_2$$

- initial vector

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = M \begin{pmatrix} x \\ x' \end{pmatrix}_s = \alpha_1 \lambda_1 Y_1 + \alpha_2 \lambda_2 Y_2$$

- after a turn

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C\cdot N} = M^N \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s = \alpha_1 \lambda_1^N Y_1 + \alpha_2 \lambda_2^N Y_2$$

- after N turns

For the motion to be stable

$$|\lambda_1|, |\lambda_2| \leq 1$$

Stability of Betatron Motion [2]

Matrices T and M are Wronskians => det(T) – constant.
det(T) = det(M) = 1 – obtain from initial conditions

$$\lambda_1 \cdot \lambda_2 = 1.$$

$$\lambda_1 = \bar{\lambda}_2 \quad - \text{c.c.}$$

$$\Rightarrow \quad \lambda = e^{i\mu}$$

μ is the betatron phase advance per turn

$$|M - \lambda E| = 0$$

$$\lambda^2 - \lambda(m_{11} + m_{22}) + \det M = 0$$

$$\lambda = \frac{\mathrm{Tr}\,M}{2} \pm i\sqrt{1 - \left(\frac{\mathrm{Tr}\,M}{2}\right)^2} = e^{i\mu}$$

$$\cos\mu = \frac{\mathrm{Tr}\,M}{2}$$

$$-1 < \frac{\mathrm{Tr}\,M}{2} < 1 \quad - \text{Stability condition}$$

Twiss Parametrization

$M = I\cos\mu + J\sin\mu =$

I - unity matrix

$$= \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} \qquad J = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

Twiss parametrization
Because det(M)=1

$$-\alpha^2 + \beta\gamma = 1$$

Eigen Vector

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad M\begin{bmatrix} y \\ y' \end{bmatrix} = e^{\pm i\mu}\begin{bmatrix} y \\ y' \end{bmatrix}$$

$$(\cos\mu + \alpha\sin\mu)y + \beta\sin\mu\, y' = e^{\pm i\mu} \cdot y$$

Equalize ~~sism~~ sine and Cosine terms $\alpha y + \beta y' = \pm i y$

$$\frac{y'}{y} = \frac{\pm i - \alpha}{\beta}$$

$$\Rightarrow \quad Y = \begin{bmatrix} \sqrt{\beta} \\ \frac{\pm i - \alpha}{\sqrt{\beta}} \end{bmatrix} \leftarrow \text{eigenvector}$$

Evolution of Particle Coordinates at Specific Location *s*

$$x = \alpha_1 \sqrt{\beta} + \alpha_2 \sqrt{\beta}$$

$$x' = \alpha_1 \frac{i - \alpha}{\sqrt{\beta}} + \alpha_2 \frac{-i - \alpha}{\sqrt{\beta}}$$

α_1, α_2 – complex, must be

$$\alpha_1 = \alpha_2^*$$

$$\alpha_1 = A e^{i\varphi}, \quad \alpha_2 = A e^{-i\varphi}$$

$$\Rightarrow \quad x = A \cdot \sqrt{\beta} \cdot \cos\varphi$$

$$x' = -\frac{A}{\sqrt{\beta}} \left(\sin\varphi + \alpha \cdot \cos\varphi \right)$$

$\Rightarrow$ After N turns

$$x = A\sqrt{\beta} \cdot \cos(\varphi + N \cdot \mu)$$

$$x' = -\frac{A}{\sqrt{\beta}} \left(\sin(\varphi + N \cdot \mu) + \alpha \cdot \cos(\varphi + N \cdot \mu) \right)$$

μ is the betatron phase advance per turn

Courant-Snyder Ellipse At Specific Location *s*

$x = A\sqrt{\beta}\cos\psi$ $\quad\quad\quad$ $\psi = \varphi + \mu \cdot N$

$$x' = -\frac{A}{\sqrt{\beta}}\left(\alpha\cos\psi + \sin\psi\right)$$

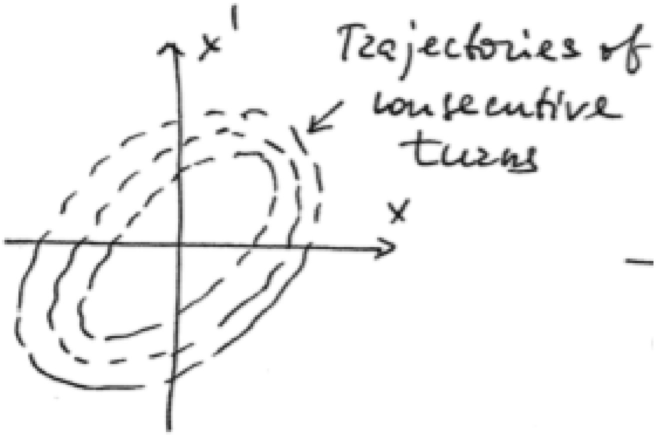
$$A\cos\psi = \frac{x}{\sqrt{\beta}}$$

$$A\sin\psi = -\left(x'\sqrt{\beta} + \frac{\alpha x}{\sqrt{\beta}}\right)$$

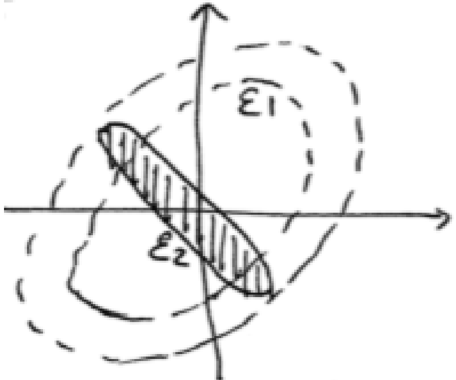
$$A^2 = A^2\cos^2\psi + A^2\sin^2\psi =$$

$$= \frac{1+\alpha^2}{\beta}x^2 + 2\alpha x x' + \beta x'^2 =$$

$$= \gamma x^2 + 2\alpha x x' + \beta x'^2 \longrightarrow \boxed{\text{invariant}}$$



$\boxed{\text{Mismatch}}$!



Area of ε_1 > Area of ε_2

However, effective area of ε_2 is larger than ε_1.

Particle Motion Along Accelerator
Equations for α and β

(1) $\left(\frac{y'}{y}\right) = \frac{\pm i - \alpha}{\beta}$ $\qquad y'' + ky = 0$

$Y_{1,2} = \begin{bmatrix} y \\ y' \end{bmatrix}$ – eigenvector

Diffirentiate left and righ parts of (1)

$$\left(\frac{y'}{y}\right)' = -k - \left(\frac{y'}{y}\right)^2 = -k - \left(\frac{\pm i - \alpha}{\beta}\right)^2$$

$$\left(\frac{\pm i - \alpha}{\beta}\right)' = -\frac{\alpha'}{\beta} - \frac{(\pm i - \alpha)}{\beta^2}\beta'$$

Equlize real and complex parts:

2) $\beta' = -2\alpha$

$\alpha' = k\beta - \frac{1+\alpha^2}{\beta}$ – system of equations for α and β

or $\frac{1}{2}\beta\beta'' + k\beta^2 - \frac{\beta'^2}{4} = 1$

Particle Motion Along Accelerator
Phase Advance

$$\frac{y'}{y} = \frac{\pm i - \alpha}{\beta} = \frac{\pm i + \beta'/2}{\beta}$$

$$\frac{dy}{y} = \pm i \frac{ds}{\beta} + \frac{1}{2}\frac{d\beta}{\beta}$$

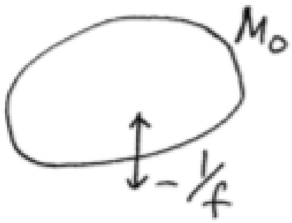
$$y = \alpha_{1,2}\sqrt{\beta}\, e^{\pm i \int \frac{ds}{\beta}}$$

$$x = A \cdot \sqrt{\beta} \cdot \cos\left(\int_{s_0}^{s_0+s} \frac{ds}{\beta(s)} + \varphi\right)$$

$$\int_{s_0}^{s_0+C} \frac{ds}{\beta} = \mu \text{ - phase advance per turn}$$

$$\nu = \frac{\mu}{2\pi} = \frac{1}{2\pi}\int_{s_0}^{s_0+C} \frac{ds}{\beta}$$ - Betatron tune

Example
Small Focusing Perturbation



Thin, weak lens added to a ring with the one-turn matrix M0
Find new betatron tune and β at the location of the lens

$$M = \overset{F}{\begin{bmatrix}1 & 0\\ -1/f & 1\end{bmatrix}} \overset{M_0}{\begin{bmatrix}\cos\mu_0+\alpha_0\sin\mu_0 & \beta_0\sin\mu_0\\ -\gamma_0\sin\mu_0 & \cos\mu_0-\alpha_0\sin\mu_0\end{bmatrix}}$$

$$= \begin{bmatrix}\cos\mu_0+\alpha_0\sin\mu_0 & \beta\sin\mu_0\\ -\frac{1}{f}(\cos\mu_0+\alpha_0\sin\mu_0)-\gamma_0\sin\mu_0 & \cos\mu_0-\alpha_0\sin\mu_0-\frac{\beta_0}{f}\sin\mu_0\end{bmatrix}$$

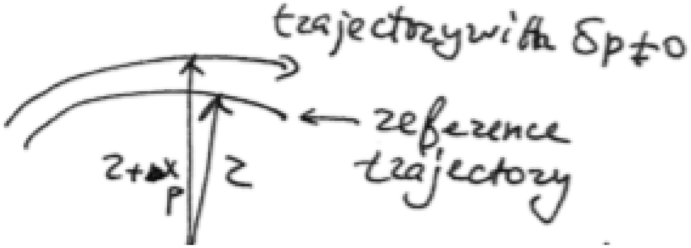
$$\cos\mu = \cos(\mu_0+\Delta\mu) = \frac{\mathrm{Tr}\,M}{2} = \cos\mu_0 - \frac{\beta_0}{2f}\sin\mu_0$$

$$\cos(\mu_0+\Delta\mu) \simeq \cos\mu_0 - \sin\mu_0\cdot\Delta\mu$$ For small $\delta\mu$

$$\Rightarrow \quad \Delta\mu = \frac{\beta}{2f}$$

$$\Rightarrow \quad \Delta\nu = \frac{\Delta\mu}{2\pi} = \frac{\beta}{4\pi f}$$

Motion of Particle With Energy Deviation



$$x'' + kx = \frac{1}{\rho}\frac{\Delta p}{p}$$

$$x = \eta(s)\,\frac{\Delta p}{p}$$

Search for solution in this form

$$\eta'' + k\eta = \frac{1}{\rho}$$

Equation for dispersion function

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \cdot \int \frac{e\sqrt{\beta(s')}}{pc} B(s') \cdot \cos\left(|\psi(s) - \psi(s')| - \pi\nu\right) ds'$$

Change in Circumference

$$\Delta C = \oint \frac{x_p}{\rho} = \frac{\Delta p}{p} \oint \frac{\eta}{\rho} ds = \alpha_p C \frac{\Delta p}{p}$$

$$\alpha_p = \frac{1}{C} \int \frac{\eta}{\rho} ds$$

$$\Rightarrow \frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p}$$

Azimuthally Symmetric Case

$$x'' + \frac{1-n}{r^2} x = \frac{1}{r} \frac{\Delta p}{p}$$

$$\eta'' + \frac{1-n}{r^2} \eta = \frac{1}{r}$$

$$\eta = \frac{r}{1-n} \qquad \left(\text{trend} \quad \eta \sim \frac{r}{\nu_x^2} \right)$$

$$\alpha_p = \frac{1}{1-n}$$

Energy-Phase Motion with RF [1]

$\omega_{rf} = h\omega$ − RF frequency $\underline{\Delta E = eU \cdot \sin\varphi}$!

- sine dependence!

h − harmonic number

φ − RF phase

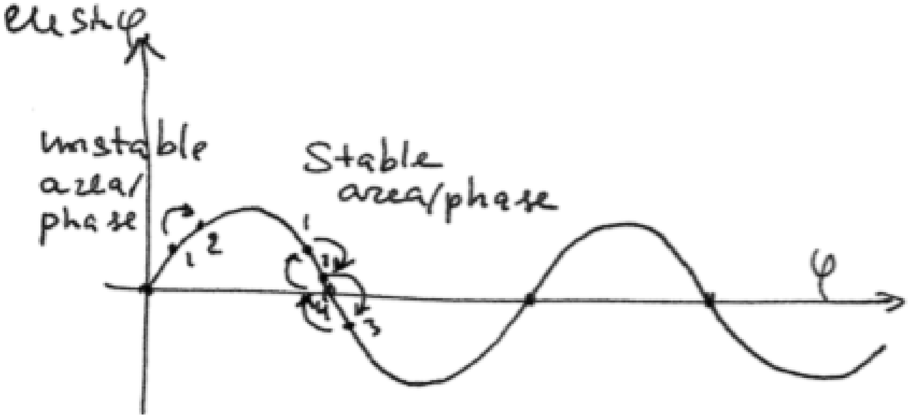
ω − revolution frequency

$T = \frac{2\pi}{\omega}$ − revolution time $T = \frac{C}{v}$

$$\Delta T = \Delta\left(\frac{C}{v}\right) = \frac{\Delta C \cdot v - \Delta v \cdot C}{v^2} = T\left(\frac{\Delta C}{C} - \frac{\Delta v}{v}\right) = T\left(\alpha_p - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

$$= T\left(\alpha_p - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

$\alpha_p - \frac{1}{\gamma^2}$ − is typically positive for electron rings (can be negative for ion and proton rings)



$\varphi_1 < \varphi_2$

means particle 1 arrives earlier than particle 2

Energy-Phase Motion with RF [2]

Rate of phase change

$$\varphi_{i+1} = \varphi_i + h\omega T\left(\alpha_p - 1/\gamma^2\right)\frac{\Delta p}{p}$$

$$\frac{d\varphi}{dt} = \frac{\Delta\varphi}{T} = h\omega\left(\alpha_p - 1/\gamma^2\right)\frac{\Delta p}{p}$$

Assumes slow
Change per turn

Rate of energy change

$$E_{i+1} = E_i + eU\cdot\sin\varphi - W_0$$

$$E_s = E_s + eU\cdot\sin\varphi_s - W_0$$

W_0 is energy loss per turn to radiation

For synchronous phase, φ_s, W_0 is exactly compensated by energy gain

$$\varepsilon_{i+1} = \varepsilon_i + eU(\sin\varphi - \sin\varphi_s) \qquad \varepsilon = E - E_s$$

$$\frac{d\varepsilon}{dt} = \frac{\Delta\varepsilon}{T} = \frac{eU}{T}(\sin\varphi - \sin\varphi_s)$$

$$d\varepsilon = pv\frac{dp}{p}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\Delta p}{p}\right) = \frac{eU}{pvT}(\sin\varphi - \sin\varphi_s)$$

Equations of energy-phase motion

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\Delta p}{p}\right) &= \frac{eU}{pvT}(\sin\varphi - \sin\varphi_s) \\ \frac{d\varphi}{dt} &= h\omega\left(\alpha_p - 1/\gamma^2\right)\frac{\Delta p}{p} \end{aligned}}$$

Small Oscillations

For small deviations in phase from the synchronous phase $\Delta\varphi = \varphi - \varphi_s \ll 1$

$$\sin\varphi - \sin\varphi_s = \sin(\varphi_s + \Delta\varphi) - \sin\varphi_s \simeq \cos\varphi_s \, \Delta\varphi_s$$

$$d\varphi = d\Delta\varphi$$

$$\boxed{\begin{aligned} \frac{d\Delta\varphi}{dt} &= h\omega\left(\alpha_p - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} \\ \frac{d}{dt}\left(\frac{\Delta p}{p}\right) &= \frac{eU \cdot \cos\varphi_s}{p v T}\,\Delta\varphi \end{aligned}}$$

Equations of energy-phase motion for small amplitudes

$$\frac{d^2\Delta\varphi}{dt^2} = + \frac{h\omega^2\left(\alpha_p - 1/\gamma^2\right) eU \cos\varphi_s}{2\pi p v}\,\Delta\varphi$$

$$\omega_s^2 = -\frac{h\omega^2\left(\alpha_p - 1/\gamma^2\right) eU \cdot \cos\varphi_s}{2\pi p v}$$

- synchrotron frequency

$$\Rightarrow \quad \nu_s = \sqrt{-\frac{h\left(\alpha_p - 1/\gamma^2\right) eU \cdot \cos\varphi_s}{2\pi p v}}$$

- normalized synchrotron tune

Hamiltonian of Energy-Phase Motion with RF

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\Delta p}{p}\right) &= \frac{eU(\sin\varphi - \sin\varphi_s)}{pvT} \\ \frac{d\varphi}{dt} &= h\omega\left(\alpha_p - 1/\gamma^2\right)\frac{\Delta p}{p} \end{aligned}}$$

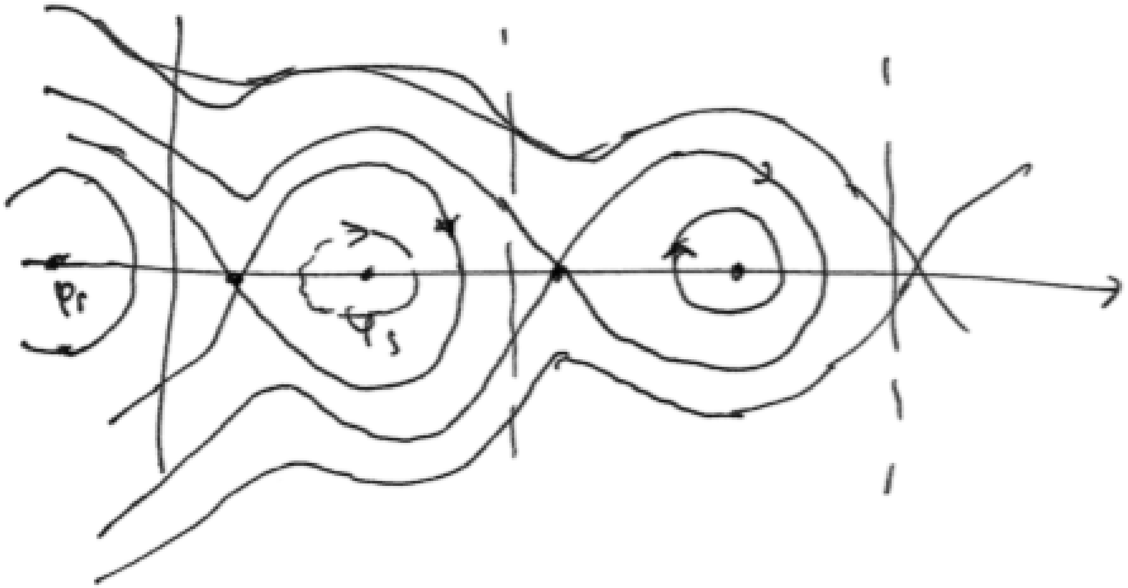
H – Hamiltonian $\quad \frac{\Delta p}{p}$ – momentum (p), φ – coordinate (q)

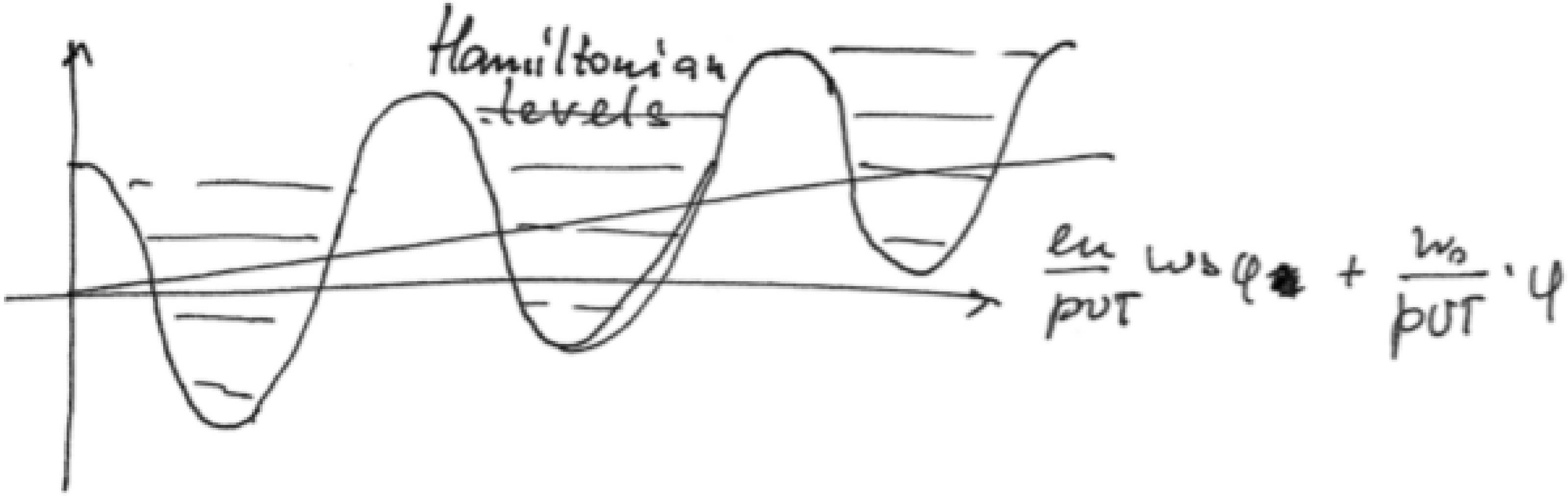
$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\boxed{H = \frac{1}{2}h\omega\left(\alpha_p - 1/\gamma^2\right)\left(\frac{\Delta p}{p}\right)^2 + \frac{eU}{pvT}\left(\cos\varphi + \sin\varphi_s \cdot \varphi\right)}$$

$$= \frac{1}{2}h\omega\left(\alpha_p - \frac{1}{\gamma^2}\right)\left(\frac{\Delta p}{p}\right)^2 + \frac{eU}{pvT}\cdot\cos\varphi + \frac{W_0}{pvT}\cdot\varphi$$

Energy-Phase Motion with RF with Arbitrary Amplitudes

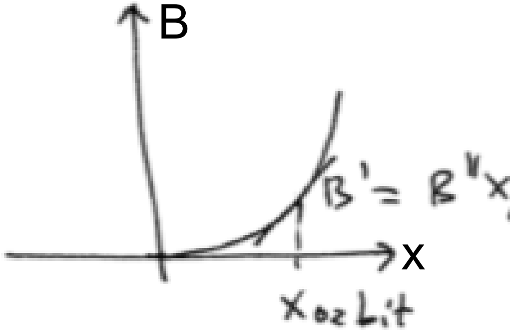




Chromatism of Betatron Oscillations [1]

Chromatism of Betatron oscillations:

$$x'' + kx = 0$$



Quad:

$$k = \frac{eB'}{pc}$$

$$B = \underset{\text{dipole}}{\underset{\uparrow}{B_0}} + \underset{\text{quad}}{\underset{\uparrow}{B'}}x + \frac{1}{2}\underset{\text{sextupole}}{\underset{\uparrow}{B''}}x^2 \quad \text{— Taylor series.}$$

$$\Rightarrow \left(\frac{dB}{dx}\right)_{x=\text{orbit}} = B' + B''x_{\text{orbit}} \qquad x_{\text{orbit}} = \eta\frac{\delta p}{p}$$

$$k = \frac{e(B' + B''x_0)}{cp(1 + \delta p/p)} \xrightarrow{\text{To the 1st order}} \frac{e}{cp}\left(\underset{k_0}{\underset{\uparrow}{B'}} + (B''\eta - \underset{\Delta k}{\underset{\uparrow}{B'}})\frac{\delta p}{p}\right)$$

Chromatism of Betatron Oscillations [2]

$$k = \frac{e(B' + B'' x_0)}{cp(1+\delta p/p)} \xrightarrow[\text{1st order}]{\text{To the}} \frac{e}{cp}\Big(\underset{\overset{\uparrow}{k_0}}{B'} + \underset{\overset{\uparrow}{\Delta k}}{(B''\eta - B')}\frac{\delta p}{p}\Big)$$

$\Delta \nu = \frac{\Delta\mu}{2\pi} = \frac{1}{4\pi}\frac{\beta}{f}$ – thin lense

$$\Rightarrow \quad \Delta\nu = \frac{1}{4\pi}\frac{\delta p}{p}\oint e\frac{B''\eta - B'}{pc}\cdot\beta\, ds$$

If $B''=0$, $\Delta\nu / \frac{\delta p}{p} < 0$. Tune spread can be very large

Tune spread can be compensated if sextupoles added next to quads as

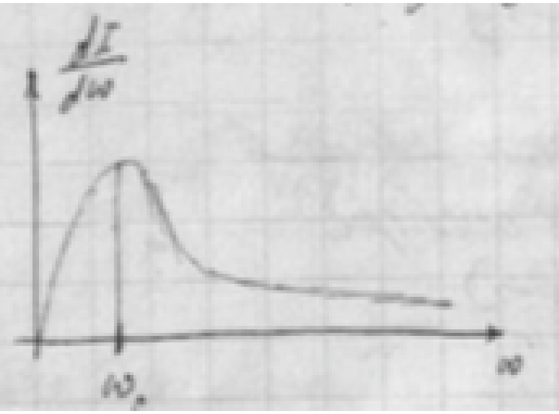
$$\beta_s \boxed{B''\eta\cdot l_{sext} = B'\cdot l_{quad}}\,\beta_q$$

Beneficial to install Sextupoles
At locations with a large
beta function

Appendix 2

Quantum Excitation of Radiation and Synchrotron Integrals

Quantum Nature of Synchrotron Radiation

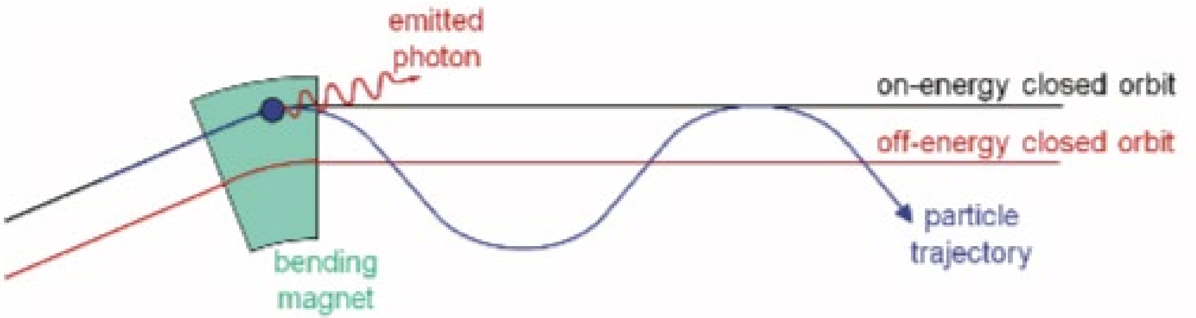


Number of photons emitted per turn $N \approx \alpha\gamma = \frac{\gamma}{137}$

α- is the fine-structure constant

Statistical emission of a quantum appears as a change in an equilibrium orbit by recoil, causing oscillations around that new orbit - increases betatron oscillations.

Multiple emissions behave like Brownian motion causing diffusion and increase of emittance.



Quantum oscillations ultimately limit the equilibrium emittance.

The equilibrium emittance is defined by the damping rate and by the growth rate caused by random emissions of light quanta.

Quantum Statistics of Synchrotron Emission

Parameter	Value
Mean photon energy, $\langle\epsilon\rangle$	$\frac{8}{15\sqrt{3}}\epsilon_{crit}$
RMS photon energy, $\langle\epsilon^2\rangle$	$\frac{11}{27}\epsilon_{crit}^2$
Total photon flux, $\dot{N}_{ph}$	$\frac{15\sqrt{3}}{8}\frac{P_\gamma}{\epsilon_{crit}}$
Product, $\dot{N}_{ph}\langle\epsilon^2\rangle$	$\frac{55}{24\sqrt{3}}P_\gamma\epsilon_{crit} = \frac{55}{24\sqrt{3}}\hbar c^2 r_e mc^2\frac{\gamma^7}{\lvert\rho^3\rvert}$

Quantum excitation over path length, L

$$\Delta\sigma_E^2\Big|_{quant} = \frac{55(\hbar c)^2}{48\sqrt{3}}\gamma^7\int_0^L\left(\frac{1}{\lvert\rho_x^3\rvert}+\frac{1}{\lvert\rho_y^3\rvert}\right)ds$$

Emittance increase over path length, L

$$\Delta\varepsilon_u\Big|_{quant} = \frac{55 r_e\hbar c}{48\sqrt{3}mc^2}\gamma^5\int_0^L\frac{\mathcal{H}_u}{\lvert\rho_u^3\rvert}$$

Equilibrium Lattice

Energy spread	$\left\langle d\sigma_E^2/dt\big\vert_{quant}\right\rangle_s = \left\langle d\sigma_E^2/dt\big\vert_{damp}\right\rangle_s = -2\alpha_s\sigma_E^2$	$\frac{\sigma_E^2}{E_0^2} = C_q\gamma^2\frac{\mathcal{J}_3}{2\mathcal{J}_2+\mathcal{J}_{4x}+\mathcal{J}_{4y}}$ $C_q = \frac{55\hbar c}{32\sqrt{3}mc^2} = 0.38319\,pm$
Bunch length	$\sigma_z = \frac{c\lvert\eta_{slip}\rvert}{\omega_s}\frac{\sigma_E}{E_0}$	$\sigma_z = \frac{\sqrt{2\pi}c}{\omega_0}\sqrt{\frac{-\eta_{slip}E_0}{heV_{rf}\cos\phi_s}}\frac{\sigma_E}{E_0}$
Horizontal beam emittance	$\left\langle d\varepsilon_x/dt\big\vert_{damp}\right\rangle_s = -2\alpha_x\varepsilon_x$	$\varepsilon_x\big\vert_{equ} = C_q\frac{\gamma^2}{J_x}\frac{\mathcal{J}_{5x}}{\mathcal{J}_2}$
Vertical beam emittance	$\left\langle d\varepsilon_y/dt\big\vert_{damp}\right\rangle_s = -2\alpha_y\varepsilon_y$ (Hor and Ver can mix due to misalignments)	$\mathcal{H}_y = 0$ (no dispersion) $\varepsilon_y\big\vert_{equ} = \frac{C_q\langle\beta_y\rangle_s}{2J_y}\frac{\langle\rho^{-3}\rangle}{\langle\rho^{-2}\rangle}$

Synchrotron Radiation Integrals

$$\mathcal{I}_1[m] = \oint \left(\frac{D_x}{\rho_x} + \frac{D_y}{\rho_y}\right) ds$$

$$\mathcal{I}_2[m^{-1}] = \oint \left(\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2}\right) ds$$

$$\mathcal{I}_3[m^{-2}] = \oint \left(\frac{1}{|\rho_x|^3} + \frac{1}{|\rho_y|^3}\right) ds$$

$$\mathcal{I}_{4u}[m^{-1}] = \begin{cases} \oint \frac{D_u}{\rho_u^3}(1 \pm 2\rho_u^2 k) ds\,, sector \\ \pm \oint 2\frac{D_u k}{\rho_u} ds, rectangular \end{cases}$$

$$\mathcal{I}_{5u}[m^{-1}] = \oint \frac{\mathcal{H}_u}{|\rho_u|^3} ds$$

$$\mathcal{I}_{6u}[m^{-1}] = \oint k^2 D_u^2 ds$$

$$\langle P_\gamma \rangle = \frac{1}{C} \oint P_\gamma\, ds = \frac{cC_\gamma}{2\pi C} E^4 \mathcal{I}_2 \quad U_0 = \frac{C_\gamma}{2\pi} E^4 \mathcal{I}_2$$

$$\alpha_u = {}^{1}/_{\tau_u} = \frac{C_\alpha}{C} E^3 \mathcal{I}_2 (1 - \mathcal{I}_{4u}/\mathcal{I}_2)$$

$$\alpha_s = {}^{1}/_{\tau_s} = \frac{C_\alpha}{C} E^3 \mathcal{I}_2 \left[2 + \left(\mathcal{I}_{4x} + \mathcal{I}_{4y}\right)/\mathcal{I}_2\right]$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} = 8.846\ 10^{-5} \frac{m}{GeV^3}$$

$$C_\alpha = 2113.1 \frac{m^2}{GeV^3\ s}$$

$$\mathcal{H}_u = \beta_u {D_u'}^2 + 2\alpha_u D_u D_u' + \gamma_u D_u^2$$

$$k = \frac{\partial B_x}{\partial y}/[B\rho] \quad u = x, y \quad \pm = \begin{matrix} x \\ y \end{matrix}$$

Robinson Theorem of the Sum of Decrements

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_s} = \frac{2W_0}{TE} = \frac{W_0}{2TE}\left(J_x + J_y + J_s\right)$$

$$J_x + J_y + J_s = 4$$

$$\frac{1}{\tau_x} = \frac{U_0}{2ET}(1 - D)$$

$$\frac{1}{\tau_s} = \frac{W_0}{2ET}(2 + D)$$

$$\frac{1}{\tau_y} = \frac{W_0}{2ET}$$

For most modern large scale machines

$$D \approx \alpha_p \frac{\bar{R}}{r} \ll 1$$

$\bar{R}$ is the average machine radius
α_p is the compaction factor
r is the magnet radius