PHY 841, HW Solutions
Chapter 1

1. $$s = (m + (m+T))^2 - \left[(m+T)^2 - m^2\right]$$

E^2 $\quad -p^2$

Here T = KE of beam for fixed target

$$s = 4m^2 + 2mT$$
$$= (2m + 2T_c)^2$$
$$T_c = \frac{1}{2}\sqrt{4m^2 + 2mT} - m$$
$$= m\left[\sqrt{1 + T/2m} - 1\right]$$
$$= 56.4 \text{ GeV}$$

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2. $$4m^2 + 2mT = 4(m + T_c)^2$$
$$T = \frac{2(m+T_c)^2 - 2m^2}{m}$$
$$= \frac{4mT_c + 2T_c^2}{m}$$
$$= 4T_c + \frac{2T_c^2}{m}$$
$$= 1.045 \cdot 10^5 \text{ TeV}$$

cosmic rays have been observed with more energy.

3. $$p' = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix}$$

$$= \begin{pmatrix} E\cosh\eta + p\sinh\eta \\ E\sinh\eta + p\cosh\eta \end{pmatrix}$$

$$p'^2 = E^2\cosh^2\eta + p^2\sinh^2\eta + 2pE\cosh\eta\sinh\eta$$

$$- E^2\sinh^2\eta - p^2\cosh^2\eta - 2pE\cosh\eta\sinh\eta$$

$$= E^2 - p^2 = m^2$$

4. In frame of a,

$$(\vec{r} + \vec{v}_b \Delta t) \cdot \vec{p}_b = 0$$

we want t.

$$\Delta t = -\frac{\vec{p}_b \cdot \vec{r}}{\vec{p}_b \cdot \vec{v}_b} = -\frac{\vec{p}_b \cdot \vec{r}}{|\vec{p}_b|^2 \frac{1}{E_b}}$$

$$\vec{p}_b = p_b' = p_b - p_a \frac{(p_b \cdot p_a)}{m_a^2}, \quad E_b = \frac{p_a \cdot p_b}{m_a}$$

$$= \frac{\left(p_b - p_a \frac{(p_b \cdot p_a)}{m_a^2}\right) \cdot r}{-\left(p_b - p_a \frac{(p_a \cdot p_b)}{m_a^2}\right)^2 \frac{m_a}{p_b \cdot p_a}}$$

$$\Delta t = \frac{-\left[p_b \cdot r - p_a \cdot r \frac{(p_a \cdot p_b)}{m_a^2}\right](p_a \cdot p_b)}{m_a \left[m_b^2 - \frac{(p_a \cdot p_b)^2}{m_a^2}\right]}$$

$$t = t_0 + \Delta t, \quad t_0 = \frac{p_a \cdot r}{m_a}$$

$$t = \frac{p_a \cdot r}{m_a} - \frac{\left[p_b \cdot r - p_a \cdot r \frac{(p_a \cdot p_b)}{m_a^2}\right](p_a \cdot p_b)}{m_a \left(m_b^2 - (p_a \cdot p_b)^2 / m_a^2\right)}$$

5. Define q so that in c.o.m. frame

$$q_0 = 0$$

$$q^\alpha = (p_1 - p_2)^\alpha - \frac{(p_1 - p_2)\cdot(p_1 + p_2)}{s}(p_1 + p_2)^\alpha$$

$$v_{rel} = \frac{|\vec{q}|}{2E_1'} + \frac{|\vec{q}|}{2E_2'} = \frac{E_1' + E_2'}{2E_1' E_2'} |\vec{q}|$$

where E_a', E_b' are energies in c.o.m. frame

$$E_1' = \frac{P\cdot p_1}{\sqrt{s}}, \quad E_2' = \frac{P\cdot p_2}{\sqrt{s}}$$

$$E_1' = \frac{m_1^2 + p_1\cdot p_2}{\sqrt{s}}, \quad E_2' = \frac{m_2^2 + p_1\cdot p_2}{\sqrt{s}}$$

$$|\vec{q}|^2 = -q^2 = -(p_1 - p_2)^2 + \frac{\left[(p_1 - p_2)\cdot(p_1 + p_2)\right]^2}{s}$$

$$= -m_1^2 - m_2^2 + 2p_1\cdot p_2 + \frac{(m_1^2 - m_2^2)^2}{s}$$

$$s|\vec{q}|^2 = s(s - 2m_1^2 - 2m_2^2) + m_1^4 + m_2^4 - 2m_1^2 m_2^2$$

$$= s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2$$

$$v_{rel}^2 = \frac{1}{4E_1'^2 E_2'^2}\left[s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2\right]$$

$$v_{rel} = \frac{s}{2(P\cdot p_1)(P\cdot p_2)}\sqrt{s^2 + m_1^4 + m_2^4 - 2(sm_1^2 + sm_2^2 + m_1^2 m_2^2)}$$

or another form,

$$s = m_1^2 + m_2^2 + 2 p_a \cdot p_b$$

$$v_{rel} = \frac{s}{2(p_1 \cdot k)(p_2 \cdot k)} \sqrt{\begin{aligned} & m_1^4 + m_2^4 + 4(p_1 \cdot p_2)^2 + 4(p_1 \cdot p_2)(m_1^2 + m_2^2) \\ & + 2m_1^2 m_2^2 - 2(m_1^2 + m_2^2 + 2p_1 \cdot p_2)\cdot(m_1^2 + m_2^2) \\ & + m_1^4 + m_2^4 - 2m_1^2 m_2^2 \end{aligned}}$$

$$= \frac{s}{2(p_1 \cdot k)(p_2 \cdot k)} \sqrt{-4 m_1^2 m_2^2 + 4(p_1 \cdot p_2)^2}$$

$$= \frac{s}{(k \cdot p_1)(p \cdot p_2)} \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

6.a) $\boxed{E = 1}$

$$A\gamma + B\gamma + C + D\gamma^2 + 1 = \gamma$$

$$-A\gamma v - D\gamma^2 v = -\gamma v$$

$$B\gamma v + D\gamma^2 v = -\gamma v$$

$$-D\gamma^2 v^2 + 1 = \gamma$$

note $\gamma^2 = \gamma^2 v^2 + 1$, $\gamma^2 v^2 = \gamma^2 - 1$

$$-D(\gamma^2 - 1) = \gamma$$

$$\boxed{D = \frac{-(\gamma - 1)}{\gamma^2 - 1} = \frac{-1}{\gamma + 1}}$$

$$A = 1 - D\gamma = \frac{\gamma + 1 + \gamma}{\gamma + 1} = \boxed{\frac{2\gamma + 1}{\gamma + 1} = A} \boxed{= 2 - \frac{1}{\gamma + 1}}$$

$$B = \frac{\gamma}{\gamma + 1} - 1 = \boxed{\frac{-1}{\gamma + 1} = B}$$

$$C = \gamma - 1 + \frac{\gamma^2 + \cancel{\gamma} - 2\gamma^2 - \cancel{\gamma}}{\gamma + 1} = \frac{-1}{\gamma + 1}$$

$$\mathcal{L}^{\alpha\beta} = g^{\alpha\beta} + \frac{-u^\alpha u'^\beta - u'^\alpha u^\beta - u^\alpha u^\beta - u'^\alpha u'^\beta}{1 + u \cdot u'} + 2u^\alpha u'^\beta$$

$$= g^{\alpha\beta} - \frac{(u^\alpha + u'^\alpha)(u^\beta + u'^\beta)}{1 + u \cdot u'} + 2u^\alpha u'^\beta$$

6b) $$\left\{ g^{\alpha\beta} + 2u^{\alpha}u'^{\beta} - \frac{(u^{\alpha}+u'^{\alpha})(u^{\beta}+u'^{\beta})}{1+u\cdot u'} \right\} u'_{\beta}$$

$$= u'^{\alpha} + 2u^{\alpha} - \frac{(u^{\alpha}+u'^{\alpha})(1+u\cdot u')}{(1+u\cdot u')}$$

$$= u^{\alpha} \quad \checkmark$$

6c) $$\left\{ g^{\alpha\beta} + 2u^{\alpha}u'^{\beta} - \frac{(u^{\alpha}+u'^{\alpha})(u^{\beta}+u'^{\beta})}{(1+u\cdot u')} \right\}$$

$$\left\{ g_{\beta}{}^{\gamma} + 2u'_{\beta}u^{\gamma} - \frac{(u_{\beta}+u'_{\beta})(u^{\gamma}+u'^{\gamma})}{1+u\cdot u'} \right\}$$

$$= g^{\alpha\gamma} + \cancel{2u^{\alpha}u'^{\gamma}} - \cancel{\frac{(u^{\alpha}+u'^{\alpha})(u^{\gamma}+u'^{\gamma})}{1+u\cdot u'}}$$

$$+ \cancel{2u^{\alpha}u'^{\gamma}} + 4u^{\alpha}u^{\gamma} - 2u^{\alpha}(\cancel{u^{\gamma}}+\cancel{u'^{\gamma}})$$

$$- \cancel{\frac{(u^{\alpha}+u'^{\alpha})(u^{\gamma}+u'^{\gamma})}{1+u\cdot u'}} - 2u^{\alpha}(\cancel{u^{\gamma}}+\cancel{u^{\gamma}})$$

$$+ 2\,\cancel{\frac{(u^{\alpha}+u'^{\alpha})(u^{\gamma}+u'^{\gamma})}{1+u\cdot u'}}$$

$$= g^{\alpha\gamma}$$