PHY 841

Chapter 2

Solutions

1)

$$S = -\sum_a m \int dt \sqrt{\left(\frac{dt'_a}{dt}\right)^2 - \left(\frac{d\vec{r}'_a}{dt}\right)^2}$$

$$\delta S = -\sum m_a \int dt \frac{\left(\frac{dt'}{dt}\right) \delta \frac{dt'}{dt}}{\sqrt{}}$$

$$= +\sum m_a \int dt \quad \delta t \left\{ \frac{d}{dt} \frac{(dt'/dt)}{\sqrt{\left(\frac{dt'}{dt}\right)^2 - \left(\frac{d\vec{r}'_a}{dt}\right)^2}} \right\}$$

$$dt'/dt = 1$$

$$\delta S = 0$$

$$\therefore \sum_a \frac{m_a}{\sqrt{1 - v_a^2}} = \text{const.}$$

2 a) $\mathcal{L} = -m\sqrt{1-v^2} + eEx$

$$m\frac{d}{dt}\gamma\vec{v} = eE\,\hat{x}$$

$$p_x = eEt$$

b) $p_y = p_{0y}$

$$H = m\gamma v^2 + m/\gamma - eEx$$

$$= m\gamma\left(v^2 + (1-v^2)\right) - eEx$$

$$= m\gamma - eEx$$

$$= \sqrt{m^2 + e^2E^2t^2 + p_y^2} - eEx$$

$$v_x = \frac{p_x}{E_k}$$

$$= \frac{eEt}{\sqrt{m^2+e^2E^2t^2+p_y^2}}$$

$$dx = \frac{eEt\,dt}{\sqrt{\varepsilon_0^2 + e^2E^2t^2}}$$

$$x = \frac{\sqrt{\varepsilon_0^2 + e^2E^2t^2}}{eE} + x_0$$

$$- \frac{\varepsilon_0}{eE}$$

c) $v_y = \dfrac{p_y}{\sqrt{m^2+p_y^2+e^2E^2t^2}}$

$\mathcal{E}_0^2 = m^2+p_y^2$

$dy = p_y \int \dfrac{dt}{\sqrt{\mathcal{E}_0^2+e^2E^2t^2}}$,

$y = \dfrac{p_y}{\mathcal{E}_0} \dfrac{\mathcal{E}_0}{eE} \int \dfrac{d\sinh\eta}{\sqrt{1+\sinh^2\eta}}$,

$\sinh\eta = \dfrac{eEt}{\mathcal{E}_0}$

$y = \dfrac{p_y}{eE}\eta$

$$y = \frac{p_y}{eE}\sinh^{-1}\frac{eEt}{\mathcal{E}_0}$$

d) $x = \dfrac{1}{eE}\sqrt{\mathcal{E}_0^2+e^2E^2t^2} - \dfrac{\mathcal{E}_0}{eE} + x_0$

$= \dfrac{1}{eE}\sqrt{\mathcal{E}_0^2+\mathcal{E}_0^2\sinh^2\left(\dfrac{eEy}{p_y}\right)} + x_0$

$= \dfrac{\mathcal{E}_0}{eE}\left(\cosh\left(\dfrac{eEy}{p_y}\right)-1\right)+x_0$

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e) $x \cong \dfrac{\mathcal{E}_0}{eE}\left[\dfrac{1}{2}\dfrac{e^2E^2y^2}{p_y^2}\right] + x_0$

$y \cong \dfrac{p_y}{eE}\dfrac{eEt}{\mathcal{E}_0} = \dfrac{p_y t}{\mathcal{E}_0}$

$x = x_0 + \dfrac{\mathcal{E}_0}{2eE}\dfrac{e^2E^2}{p_y^2}\dfrac{p_y^2}{\mathcal{E}_0^2}t^2 = x_0 + \dfrac{eEt^2}{2\mathcal{E}_0}$ ✓

matches non-rel.

e)   As p_y / m << 1

3 a)

Let $\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x}) = Bx\hat{y} - \nabla\frac{Bxy}{2}$

$\Lambda = -Bxy/2$

b) $\vec{A} = \frac{B\rho\hat{\varphi}}{2}$

$\vec{\nabla}\times\vec{A} = \hat{z}\left(\frac{\partial A_\varphi}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi} + \frac{1}{\rho}A_\varphi\right) = B\rho$

$v^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2$

$\mathcal{L} = -m\sqrt{1-v^2} + \rho^2\dot{\varphi}\frac{eB}{2}$

$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m\gamma\rho^2\dot{\varphi} + \frac{\rho^2 eB}{2}, \quad \frac{\partial\mathcal{L}}{\partial\varphi} = 0$

c) $m\gamma\rho^2\dot{\varphi}$ = constant $= l$ if ρ = constant

$\frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\gamma\dot{\rho}, \quad \frac{\partial\mathcal{L}}{\partial\rho} = e\rho\dot{\varphi}B + m\gamma\rho\dot{\varphi}^2$

$\dot{\rho} = \emptyset$ solution requires

$\dot{\varphi} = \frac{eB}{m\gamma}$ ✓

4 a) Lagrange's eq.s

$$\frac{d}{dt}(m\gamma v) = F/\gamma$$

$$\frac{dm}{dt} = -vF$$

$$\frac{d}{dt}(m\gamma v) = m\frac{d}{dt}(\gamma v) - \gamma v^2 F = \frac{F}{\gamma}$$

$$m\frac{d}{dt}(\gamma v) = \frac{F}{\gamma}(1+\gamma^2 v^2) = \gamma F$$

$$\gamma v = \frac{v}{\sqrt{1-v^2}}$$

$$\frac{d}{dt}\gamma v = \frac{dv}{dt}(\gamma + \gamma^3 v^2) = \frac{dv}{dt}\gamma^3$$

$$\frac{dv}{dt} = \frac{1}{\gamma^3}\frac{d}{dt}\gamma v = \frac{1}{\gamma^3}\frac{\gamma F}{m} = \frac{F}{m\gamma^2}$$

$$\frac{dv}{dt} \doteq (1-v^2)\frac{F}{m(t)} = (1-v^2)\, a\frac{m_0}{m}$$

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b) Set $a \equiv F/m_0$, $m = m_0 \cos at$

$$v = \sin at$$

$$\frac{dv}{dt} \stackrel{?}{=} (1-v^2)\, a\frac{m_0}{m}$$

$$a\cos at \stackrel{?}{=} (1-\sin^2 at)\, a\frac{m_0}{m_0\cos at}$$

$$= a\cos at \checkmark$$

$$x = \int dt\, v(t) = \frac{1}{a}(1-\cos at)$$

c) $0 = m_0 \cos at \qquad a = F/m_0$

$$t_{max} = \frac{\pi}{2a} = \frac{m_0 \pi}{2F}$$

d) $X(t_{max}) = \frac{1}{a}(1 - \cos a t_{max}) = \frac{1}{a}$

e) $v\, t_{max} = \sin a t_{max} = 1 = c$

f) $u_x(t_{max}) = \infty$