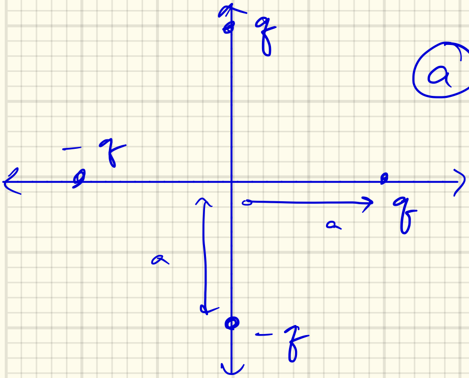
PHY 841 HW 5

Solutions

1. Consider four charges $+q, +q, -q, -q$ at the Cartesian positions $(a, 0, 0)$, $(0, a, 0)$, $(-a, 0, 0)$, $(0, -a, 0)$ respectively.

   (a) Find the dipole moments p_i and the quadrupole tensor Q_{ij} for all i and j.

   (b) Find all $q_{\ell m}$ for all $\ell \leq 2$.



(a) $p_x = p_y = qa$

$$Q_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) $q_{2m} = \emptyset$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\left(p_x - i p_y\right) = -\sqrt{\frac{3}{8\pi}}\,(1-i)\,qa$$

$$q_{10} = -\sqrt{\frac{3}{4\pi}}\,p_z = \emptyset$$

$$q_{1,-1} = \sqrt{\frac{3}{8\pi}}\,(1+i)\,qa = (-1)^m q_{11}^{*}$$

2. Consider a charge distribution with $q_{21} = q_{2-1}$=some imaginary number iQa^2, see definitions in Eq. (5.15). Draw a figure where you place a minimum number of discrete charges that reproduces the given q_{21} and q_{2-1}, while having all other $q_{ij} = 0$ for $\ell \leq 2$.

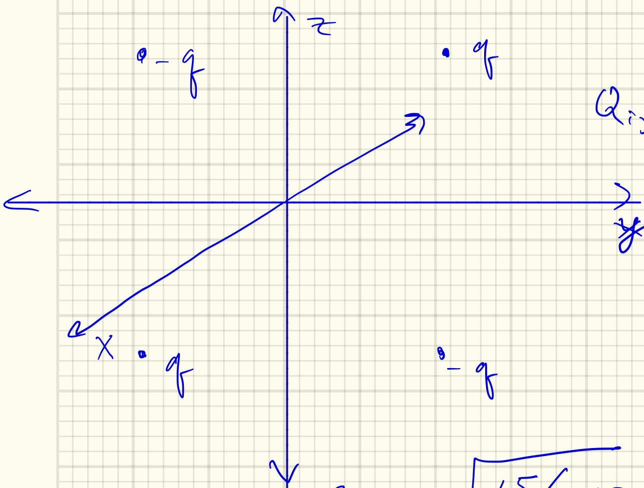
   (a) Provide the positions and find the individual charges, all of which are $\pm q$, in terms of Q. Only place charges on a lattice where the step size is a, i.e. at positions ia, ja, ka, where i, j, k are integers.

   (b) In terms of the magnitude of the individual charges, q, and the lattice spacings a, find the potential as a function of r, θ and ϕ.

a) $q_{21} = -\sqrt{15/72\pi}\,(Q_{xz} - iQ_{yz})$

Because q_{21} is imaginary you need configuration where $Q_{yz} \neq 0$ and all other $Q_{ij} = 0$.

Try this.



$$Q_{ij} = qa^2 \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

$$Qa^2 = \sqrt{15/72\pi} \cdot 4qa^2$$

$$q = \sqrt{\frac{6\pi}{5}}\,Q$$

positions = $\underset{q}{(0,a,a)}$, $\underset{-q}{(0,a,-a)}$, $\underset{-q}{(0,-a,a)}$, $\underset{q}{(0,-a,-a)}$

b)

$$V(\vec{r}) = \sum_{\ell m} \frac{4\pi}{(2\ell+1)} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}}.$$

$$Y_{2,\pm 1} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi},$$

$$q_{21} = iQa^2 \quad , \quad q_{2-1} = -(q_{21})^* = q_{21}$$

$$V = \frac{4}{5\pi} \frac{Qa^2}{r^3} \left\{ iY_{21} + iY_{2-1} \right\}$$

$$= \frac{4}{5\pi} \frac{Qa^2}{r^3} \left\{ -i\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\varphi} + i\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{-i\varphi} \right\}$$

$$= \frac{8}{5\pi} \frac{Qa^2}{r^3} \sqrt{\frac{15}{8\pi}} \left\{ \sin\theta\cos\theta\sin\varphi \right\}$$

$$= \frac{4}{5\pi} \frac{Qa^2}{r^3} \sqrt{\frac{15}{8\pi}} \sin 2\theta \sin\varphi$$

$$Qa^2 = 4qa^2\sqrt{15/(72 \cdot \pi)} = \sqrt{\left(5/24\pi\right) \cdot 16}\, qa^2$$

$$= \sqrt{10/3\pi}\, qa^2$$

$$V = \frac{4}{5\pi} \frac{qa^2}{r^3} \sqrt{\frac{10}{3\pi} \frac{15}{8\pi}} \sin 2\theta \sin\varphi$$

$$= \frac{4}{5\pi^2} \frac{qa^2}{r^3} \frac{5}{2} \sin 2\theta \sin\varphi$$

$$= \frac{2}{\pi^2} \frac{qa^2}{r^3} \sin 2\theta \sin\varphi$$

3. Consider a simple model of an atom being a particle of charge e that moves in a three-dimensional harmonic oscillator with effective spring constant k. A constant electric field E_0 is added.

   (a) What is the magnitude of the induced dipole moment p?

   (b) What is the change in total energy of the charge due to the introduction of the field? Give answer in terms of p and E_0.

a) $V = \frac{1}{2}kx^2 - eE_0 x$

$= \frac{1}{2}k\left(x - \frac{eE_0}{k}\right)^2 - \frac{1}{2}\frac{e^2E_0^2}{k}$

$\delta x = \frac{eE_0}{k}$, $p = e\,\delta x = \frac{e^2E_0}{k}$

b) $\delta E = -\frac{1}{2}e^2E_0^2/k$

$= -\frac{1}{2}pE_0$

difference with $-\vec{p}\cdot\vec{E}$ is change in energy in h.o. potential.

4. Consider a point charge q at $\vec{r}' = a\hat{z}$.

(a) Find the moments, $q_{\ell m}$ defined in Eq. (5.13), for all ℓ and m, defining the moments around the origin

(b) Show that the potential calculated with $q_{\ell m}$ using Eq. (5.14) matches Eq. (5.8) for the case where the charge is along the z axis.

(c) Show that for the case where $\Phi(\vec{r})$ is evaluated with $\vec{r}$ lying along the z-axis that the sum becomes $q/(r-a)$.

a) $\rho = q\,\delta^3(\vec{r} - a\hat{z})$

$$q_{\ell m} = q\, Y_{\ell m}(0,0)\, a^{\ell}$$

$$= 0, \quad m \neq 0$$

$$Y_{\ell m=0}(\theta, \phi) = \frac{P_\ell \sqrt{2\ell+1}}{\sqrt{4\pi}}$$

$$q_{\ell 0} = q\, a^{\ell} \sqrt{\frac{2\ell+1}{4\pi}}$$

b) $$V(\vec{r}) = \sum_{\ell, m} \frac{4\pi}{(2\ell+1)}\, q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}$$

$$V(\vec{r} = r\hat{z}) = \sum_{\ell} \frac{\cancel{4\pi}}{\cancel{(2\ell+1)}} \frac{a^{\ell}}{\cancel{4\pi}} \frac{1}{r^{\ell+1}}$$

$$= \frac{1}{r} + \frac{a}{r^2} + \frac{a^2}{r^3} + \frac{a^3}{r^4} + \cdots$$

c) $$\frac{1}{r-a} = \frac{1}{r} + \frac{a}{r^2} + \frac{a^2}{r^3} \cdots$$

$$= \frac{1}{r} \frac{1}{1-\frac{a}{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \quad ✓$$

5. Any function that can be written as a sum over Cartesian polynomials of order $\leq \ell$, i.e.,

$$F(x, y, z) = \sum_{\ell_x+\ell_y+\ell_z\leq\ell} A_{\ell_x\ell_y\ell_z} x^{\ell_x} y^{\ell_y} z^{\ell_z},$$

can be expressed as a sum of spherical harmonics with order $\leq \ell$,

$$F = \sum_{\ell', m', \ell'\leq\ell} A_{\ell'm'}(r) Y_{\ell'm'}(\theta, \phi).$$

Using this fact prove that the multipole moments of order $\leq \ell$, for the case when all moments $q_{\ell'm'}$ vanish for $\ell' < \ell$, are unaffected by a translation of the origin, and that the higher moments, $> \ell$, are affected by this change of the coordinate system. This means that the dominant multipole is unaffected by a translation of the coordinate system.
Hint: Using the definition of the moments, Eq. (5.14), replace $\rho(\vec{r})$ with $\rho(\vec{r} + \vec{a})$, then express the new charge density as a Taylor expansion in powers of a.

$$q_{\ell m} = \int d^3r\, \rho(\vec{r})\, \underbrace{r^\ell Y_{\ell m}(\theta, \varphi)}$$

Can be expressed as
$\sum A_{\ell_x\ell_y\ell_z} x^{\ell_x} y^{\ell_y} z^{\ell_z}$, $\ell_x+\ell_y+\ell_z = \ell$

$$\rho(\vec{r}) \to \rho(\vec{r} + \vec{a}) = \rho(\vec{r}) + a_i \partial_i \rho + \tfrac{1}{2} a_i a_j \partial_i \partial_j \rho + \cdots$$

if you wps ρ, the integrate by parte,

$$\Delta q_{\ell m} = \int d^3r\, \rho(\vec{r}) \underbrace{\left[-a_i \partial_i \cdots\right] r^\ell Y_{\ell m}}$$

will be sum of
$\underbrace{\sum B_{\ell_x,\ell_y\ell_z} x^{\ell_x} y^{\ell_y} z^{\ell_z} \quad \ell_x+\ell_y+\ell_z < \ell\,!}$

can then be written as

$$\Delta q_{\ell m} = \int d^3r\, \rho(\vec{r}) \Big[\sum_{\substack{\ell', m' \\ \ell'<\ell}} C_{\ell' m'} Y_{\ell' m'}\Big] = 0\,!$$

if and only if $q_{\ell' m'} = 0$ for $\ell' < \ell$