PHY 841 _ HW7

Solutions

1. Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\nu\mu} - \frac{\lambda}{2}(\partial \cdot A)^2.$$

Here, the action is $S = (1/4\pi)\int d^4x\,\mathcal{L}$. The extra term here (proportional to λ) is known as the gauge-fixing term. The equations of motion for the field are:

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\gamma} = \frac{\partial \mathcal{L}}{\partial A_\gamma}, \tag{7.35}$$

and the stress-energy tensor is:

$$T^{\alpha\beta} = \partial^\alpha A_\gamma \frac{\partial \mathcal{L}}{\partial \partial_\beta A_\gamma} - g^{\alpha\beta}\mathcal{L}. \tag{7.36}$$

(a) Show that the Lagrangian becomes (after integrating the action by parts)

$$\mathcal{L} = \frac{1}{2}A^\mu \partial^2 A_\mu - \frac{(\lambda-1)}{2}(\partial \cdot A)^2 = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \frac{(\lambda-1)}{2}(\partial \cdot A)^2.$$

(b) Find the equations of motion for A for arbitrary λ. Note that in the Feynmann gauge, $\lambda = 1$, the equations of motion do not mix different components of A, and that in the Landau gauge, $\lambda = \infty$, the Lorentz gauge is effectively enforced.

(c) Solve for the stress-energy tensor for arbitrary λ, and show that when you enforce the equations of motion you will get a result independent of λ (You may need to integrate by parts).

(d) Express T_{00} and then re-express in terms of $\vec{E}$ and $\vec{B}$.

(e) Show that $T^\alpha{}_\alpha = 0$.

a) $\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\nu\mu} - \frac{\lambda}{2}(\partial\cdot A)^2$

$= \frac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu - \frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \frac{\lambda}{2}\partial_\mu A^\mu \partial^\nu A_\nu$

The 1st term can be manipulated by integrating by parts twice, switching the place of ∂^μ & ∂_ν:

$\mathcal{L} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \frac{(\lambda-1)}{2}(\partial_\mu A^\mu)(\partial^\nu A_\nu)$

$= -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \frac{\lambda-1}{2}(\partial\cdot A)^2$

$= \frac{1}{2}A^\nu \partial^2 A_\nu - \frac{(\lambda-1)}{2}(\partial\cdot A)^2$ integrate by parts

b) Eq.s of motion

$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$-\partial_\mu(\partial^\mu A^\nu) - (\lambda-1)\partial_\mu g^{\mu\nu}\partial\cdot A$

$\partial^2 A^\nu = (1-\lambda)\partial^\nu(\partial\cdot A)$

$\partial^2 A^\nu = 0$ for $\lambda = 1$

c)

$$T^{\alpha\beta} = -\partial^\alpha A^\gamma \partial^\beta A_\gamma + \frac{g^{\alpha\beta}}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu)$$

$$T^{00} = -\dot{A}_0^2 + |\dot{\vec{A}}|^2 + \frac{1}{2}\dot{A}_0^2 - \frac{1}{2}|\dot{\vec{A}}|^2$$

$$- \frac{1}{2}(\nabla A_0)^2 + \frac{1}{2}(\partial_i A_j)^2$$

Aside: $\partial_i A_j \partial_i A_j = (\vec{\nabla}\times\vec{A})^2 + \partial_i A_j \partial_j A_i$

$= (\vec{\nabla}\times\vec{A})^2 + (\vec{\nabla}\cdot\vec{A})(\vec{\nabla}\cdot\vec{A})$ int. by parts

$$T^{00} = \frac{1}{2}|\dot{\vec{A}}|^2 - \frac{1}{2}(\nabla A_0)^2 - \cancel{\frac{1}{2}\dot{A}_0^2} + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2 + \cancel{\frac{1}{2}(\vec{\nabla}\cdot\vec{A})^2}$$

$$= \frac{1}{2}|\dot{\vec{A}}|^2 - \frac{1}{2}(\nabla A_0)^2 + \frac{1}{2}(\vec{\nabla}\times\vec{A})^2$$

ASIDE

$$|\vec{E}|^2 = (\nabla A_0)^2 + \dot{\vec{A}}^2 + 2\nabla A_0 \cdot \dot{\vec{A}}$$ int. by parts

$$= (\nabla A_0)^2 + \dot{\vec{A}}^2 - 2A_0(\vec{\nabla}\cdot\dot{\vec{A}})$$ $\partial \cdot A = 0$

$$= (\nabla A_0)^2 + \dot{\vec{A}}^2 + 2A_0\,\partial_t^2 A_0$$ Eq.s of motion

$$= (\nabla A_0)^2 + \dot{\vec{A}}^2 + 2A_0\nabla^2 A_0$$ int by parts

$$= -(\nabla A_0)^2 + (\dot{\vec{A}})^2$$

$$T^{00} = (|\vec{E}|^2 + |\vec{B}|^2)/2$$ ✓

f) $A^\mu \to A^\mu + \partial^\mu \Lambda$

$\partial \cdot A \to \partial \cdot A + \partial^2 \Lambda$

to make $\partial \cdot A = 0$, one must find solution to eq.

$\partial^2 \Lambda = S(x)$, where source function is $-\partial \cdot A$

Yes, there is always such a solution!

2. Consider solutions for electro magnetic waves moving in the $\pm z$ directions which are linearly polarized in the x direction.

(a) Find the linear combination of such waves that vanishes at $z = 0$, i.e. reflecting off a conducting plane.

(b) Find the elements of the stress-energy tensor as a function of z and t.

a) $\vec{E} = E_0 \hat{x}\, e^{-i\omega t}\left(e^{ikz} - e^{-ikz}\right)/2$

$= i E_0 \hat{x}\, e^{-i\omega t}(\sin kz)$

$\mathrm{Re}\, \vec{E} = E_0 \hat{x} \sin kz \sin \omega t$

$k = \omega$

$\vec{B} = E_0 \hat{y}\, e^{-i\omega t}\left(e^{ikz} + e^{-ikz}\right)/2$

$\mathrm{Re}\, \vec{B} = E_0 \hat{y} \cos \omega t \cos kz$

b) $T_{00} = \frac{1}{8\pi} |\vec{E}^2 + \vec{B}^2|$

$= \frac{E_0^2}{8\pi}\left\{\cos^2 \omega t \cos^2 kz + \sin^2 \omega t \sin^2 kz\right\}$

$T_{0z} = \frac{E_x B_y}{8\pi} = \frac{E_0^2}{8\pi} \sin kz \cos kz \cos \omega t \sin \omega t$

$= \frac{E_0^2}{32\pi} \sin 2kz \sin 2\omega t$

$T_{zz} = \frac{1}{8\pi} |\vec{E}^2 + \vec{B}^2| = T_{00}$

$= (E_0^2/8\pi)\left[\cos^2 \omega t \cos^2 kz + \sin^2 \omega t \sin^2 kz\right]$

$T_{xx} = \frac{1}{8\pi}\left[E_0^2 (\cos^2 \omega t \cos^2 kz + \sin^2 \omega t \sin^2 kz) - 2E_0^2 \cos^2 \omega t \cos^2 kz\right]$

$= \frac{1}{8\pi} E_0^2 \left[-\cos^2 \omega t \cos^2 kz + \sin^2 \omega t \sin^2 kz\right]$

$T_{yy} = \frac{1}{8\pi} E_0^2 \left[\cos^2 \omega t \cos^2 kz - \sin^2 \omega t \sin^2 kz\right]$

$T_{zz} = T_{00} = \frac{1}{8\pi} E_0^2 \left[\cos^2 \omega t \cos^2 kz + \sin^2 \omega t \sin^2 kz\right]$

c) Show that $T^i{}_i = 0$:

$$T^i{}_j = -T_{ii}$$

By inspection,

$$T^z{}_z = -T^0{}_0$$

$$T^x{}_x = -T^y{}_y$$

so it is traceless

2. Consider a plane wave moving in the z direction according to Eq. (7.4) with $a_y = ia_x$. Taking the real part of the solution, solve for the direction of $\vec{a}$ as a function of time.

r k axis, the solutions have the forms

$$\begin{aligned} E_i(\vec{r},t) &= a_i e^{i\vec{k}\cdot\vec{r}-i\omega t}, \\ B_i(\vec{r},t) &= b_i e^{i\vec{k}\cdot\vec{r}-i\omega t}, \\ \omega &= |\vec{k}|. \end{aligned} \tag{7.4}$$

$$E_x = a_x e^{ikr - i\omega t}$$

$$E_y = i a_x e^{ikr - i\omega t}$$

$$\mathrm{Re}(\vec{E}) = \hat{x}\, a_x \cos(kr - \omega t) - \hat{y}\, a_x \sin(kr - \omega t)$$

$$\varphi = -\tan^{-1} \frac{\sin(kr - \omega t)}{\cos(kr - \omega t)}$$

$$= -(kr - \omega t)$$

angle about z axis

and position.

4. Consider a simple model of the universe where the expansion velocity for cosmological purposes is $\vec{v} = \vec{r}/t$. This corresponds to a "flat" universe with gravitational effects ignored. All matter starts at a point (the origin) and there is no acceleration for any fluid element. Observer A is moving with the source, and she records light being emitted at a time $\tau_0 = 10^5$ years after the birth of the universe, according to a clock in her pocket. A second observer, B, records the light moving past at a time $\tau = 1.4 \times 10^{14}$ years after the beginning of the universe according to a watch in his pocket. Both A and B are at rest relative to the neighboring expanding matter. If observer A records the frequency of the emitted light as being f_0, find the frequency f of the recorded light according to observer B.
Some Help: the time measured by the co-moving observer, τ, is related to the time measured by a different observer with velocity v by the relations:

$$\tau = \frac{t}{\gamma} = t\sqrt{1 - v^2} = t\sqrt{1 - r^2/t^2} = \sqrt{t^2 - r^2}.$$

Let observer A see light recorded by B at time t and distance r in her frame.

$r = vt = c(t - \tau_0)$, where v is velocity of B according to A.

$\tau_0 = t(1 - v/c)$ (leave off c here after)

$\tau = t\sqrt{1 - v^2}$

Divide eq-s,

$$\frac{\tau_0}{\tau} = \sqrt{\frac{1-v}{1+v}} = \frac{\omega}{\omega_0} = 7.14 \cdot 10^{-6}$$

5. Show that there is no solution to the conditions for the rectangular wave-guide amplitudes in Eq. (7.19) when both $\boldsymbol{E}_{0z}$ and $\boldsymbol{B}_{0z}$ are set to zero. This demonstrates that there are no solutions other than the TE and TM solutions.

itions

$$
\begin{aligned}
-i\omega B_{0x} &= ik_z E_{0y} + q_y E_{0z}, \\
-i\omega B_{0y} &= -ik_z E_{0x} - q_x E_{0z}, \\
-i\omega B_{0z} &= q_x E_{0y} - q_y E_{0x}, \\
i\omega E_{0x} &= ik_z B_{0y} + q_y B_{0z}, \\
i\omega E_{0y} &= -ik_z B_{0x} - q_x B_{0z}, \\
i\omega E_{0z} &= q_x B_{0y} - q_y B_{0x}.
\end{aligned} \tag{7.19}
$$

$$-i\omega B_{0x} = ik_z E_{0y}$$

$$-i\omega B_{0y} = -ik_z E_{0x}$$

$$0 = q_x E_{0y} - q_y E_{0x}$$

$$i\omega E_{0x} = ik_z B_{0y}$$

$$i\omega E_{0y} = -ik_z B_{0x}$$

$$0 = q_x B_{0y} - q_y B_{0x}$$

$$-i\omega B_{0x} = ik_z\left(-\frac{k_z}{\omega}\right)B_{0x}$$

$$B_{0x} = B_{0x}\cdot\frac{k_z^2}{\omega^2}$$

can't be true unless

$q_x = q_y = \emptyset$.

6. Show that Eq.s (7.23-7.26) satisfy the Maxwell relation $\nabla \times \vec{E} = -\partial_t \vec{B}$.

gives $E_z(x, y)$ by the relation,

$$E_z = \psi(x, y)e^{-i\omega t + ik_z z}. \tag{7.23}$$

One can solve the equations for ψ from the differential equation,

$$-(\partial_x^2 + \partial_y^2)\psi = -(\omega^2 - k_z^2)\psi, \tag{7.24}$$

with the boundary conditions

$$\psi(x, y)|_S = 0. \tag{7.25}$$

This boundary condition forces E_z to be zero at the surface. Once one has solved the boundary condition, the transverse components of the electric and magnetic fields can be found via,

$$\vec{E}_t(x, y) = \frac{ik_z}{(\omega^2 - k_z^2)} e^{-i\omega t + ik_z z} \nabla_t \psi(x, y), \tag{7.26}$$

$$\vec{B}_t(x, y) = \left(\frac{\omega}{k_z}\right) \hat{z} \times \vec{E}_t.$$

$$\vec{E} = e^{-i\omega t + ik_z z}\left\{\hat{z}\psi + \frac{ik_z}{\omega^2 - k_z^2}\vec{\nabla}_t\psi\right\}$$

$$\vec{B} = e^{-i\omega t + ik_z z}\left\{\frac{i\omega}{\omega^2 - k_z^2}\hat{z} \times \vec{\nabla}_t \psi\right\}$$

$$\vec{\nabla} \times \vec{E} = e^{(\cdots)}\left\{-\hat{z} \times \vec{\nabla}_t\psi - \frac{k_z^2}{\omega^2 - k_z^2}\hat{z} \times \nabla_t \psi\right\}$$

$$\partial_t \vec{B} = e^{(\cdots)}\frac{\omega^2}{\omega^2 - k_z^2}\hat{z} \times \nabla_t \psi$$

$$\left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = e^{-i\omega t + ik_z z}\left\{\frac{-\omega^2 + k_z^2 - k_z^2 + \omega^2}{\omega^2 - k_z^2}\right\}\hat{z} \times \nabla_t \psi$$

$$= 0 \quad \checkmark$$

7. Consider a circular wave-guide of radius R. Consider the lowest frequency TM solution to the generating function ψ satisfying the differential equation

$$\nabla_t^2\psi(\rho,\phi) = -\alpha^2\psi(\rho,\phi), \quad \alpha^2 = \omega^2 - k_z^2,$$

where k_z is the wavenumber for longitudinal motion.

(a) Find a solution for ψ in polar coordinates. Express answer in terms of a_1, the first zero of the Bessel function J_1.

(b) Find expressions for the electric and magnetic fields.

(c) What is the group velocity of a wave with momentum k_z.

a) $\psi = J_0(\alpha r)$, $\quad a_0 = 2.4048$

$\alpha = 2.4048/R$

$= a_0/R$

b) $E_z = C\, J_0(\alpha r)\, e^{-i\omega t + ik_z z}$

$\vec{E}_t = \dfrac{ik_z}{\omega^2 - k_z^2}(\nabla_t \psi)\, e^{-i\ldots}$

$= \dfrac{iCk_z}{\omega^2 - k_z^2}\, e^{-i\omega t + ik_z z}\left(\hat{r}\,\partial_r J_0\left(\frac{a_0}{R}r\right)\right)$

$= \dfrac{iCk_z}{(\omega^2 - k_z^2)}\, e^{-i\omega t + ik_z z}\cdot \dfrac{a_0}{R}\,\hat{r}\left(-J_1\left(\frac{a_0}{R}r\right)\right)$

$\vec{B}_t = \dfrac{-iC\omega}{(\omega^2 - k_z^2)}\, e^{-i\omega t + ik_z z}\,\dfrac{a_0}{R}\,\hat{\phi}\, J_1\left(\frac{a_0}{R}r\right)$

c) $\omega^2 = \alpha^2 + k_z^2$, $\quad \alpha = a_0/R$

$a_0 = 2.4048$

$\dfrac{d\omega}{dk} = v_g = \dfrac{k_z}{\omega}$

$= \dfrac{k_z}{\sqrt{\left(\frac{a_0}{R}\right)^2 + k_z^2}}$

8. Consider two infinite parallel plates with the plane of the plates being along the x direction and the separation being L_x, i.e. a rectangular wave guide with $L_y = \infty$. Consider a wave moving in the z direction with wave number k_z. Using the method of generating functions,

(a) Solve for the lowest frequency ~~TE~~ TM wave. Find expressions for the fields and the the group velocity.

(b) Solve for the lowest frequency ~~TM~~ TE wave. Again find expressions for the fields and the the group velocity.

a) $q_y = 0 \,, \quad \psi = c \sin(\pi x / L_x)$

$q_x = (\pi / L_x)$

$\omega^2 = \left(\frac{\pi}{L_x}\right)^2 + k_z^2$

$E_z = c\, e^{-i\omega t + i k_z z} \sin(\pi x / L_x)$

$E_x = \frac{i k_z}{\omega^2 - k_z^2}\, c\, e^{-i(\;)} \frac{\pi}{L_x} \cos\left(\frac{\pi x}{L_x}\right)$

$E_y = 0$

$B_z = B_x = 0$

$B_y = \frac{i\omega}{\omega^2 - k_z^2}\, c\, e^{-i\omega t + i k_z z} \frac{\pi}{L_x} \cos\left(\frac{\pi x}{L_x}\right)$

$v_g = \frac{k_z}{\omega} = \frac{k_z}{\sqrt{k_z^2 + (\pi/L_x)^2}}$

---

b) $B_z = c\, e^{-i\omega t + i k_z z} \cos\left(\frac{\pi x}{L_x}\right)$

$B_x = \frac{-i k_z}{\omega^2 - k_z^2}\, c\, e^{-i\omega t + i k_z z} \frac{\pi}{L_x} \sin\left(\frac{\pi x}{L_x}\right)$

$B_y = 0$

$E_x = E_z = 0$

$E_y = \frac{i\omega}{\omega^2 - k_z^2}\, c\, e^{-i\omega t + i k_z z} \frac{\pi}{L_x} \sin\left(\frac{\pi x}{L_x}\right)$

$v_g = \frac{k_z}{\sqrt{k_z^2 + (\pi/L_x)^2}}$