## Problem 1.A Solution

Starting from the equations of motions for a relativistic charged particle

$$\frac{d}{dt}(m\gamma\vec{v}) = \frac{d}{dt}(\vec{p}) = e\left(\vec{E} + \frac{1}{c}(\vec{v}\times\vec{B})\right) \quad (1.1)$$

where both the electric field and magnetic field point along the z-axis and have equal magnitudes, we obtain from the z-component of the equation of motion

$$\frac{dp_z}{dt} = eE \;\Rightarrow\; p_z = eEt \quad (1.2)$$

We now want to solve for $v_z$ as a function of time. By using the definition of the total energy of a relativistic free particle (which coincides with the kinetic energy of the charged particle in this case), we obtain the following relation between $v_z$ and $p_z$

$$E_{kin} = \gamma mc^2, p_z = \gamma m v_z \;\Rightarrow v_z = \frac{p_z c^2}{E_{kin}} \quad (1.3)$$

$$E_{kin} = \sqrt{(mc^2)^2 + (p_x(t)c)^2 + (p_y(t)c)^2 + (p_z(t)c)^2} \quad (1.4)$$

It should be noted that the quantity $(p_x(t)c)^2 + (p_y(t)c)^2$ is a constant of motion (see appendix for proof), thus from now on we let $\kappa_0^2 = (mc^2)^2 + (p_x(t)c)^2 + (p_y(t)c)^2$ define this constant value. By combining equations 1.2, 1.3 and 1.4 we obtain the following differential equation for z as a function of time

$$\frac{dz}{dt} = v_z(t) = \frac{eEc^2t}{\sqrt{\kappa_0^2 + (eEct)^2}} \Rightarrow z(t) = \frac{1}{eE}\sqrt{\kappa_0^2 + (eEct)^2} \quad (1.5)$$

To solve the equations of motion along the x and y axes we use the following change of variables to simplify the solutions and make more evident the circular motion in the xy-plane

$$\eta(t) = p_x(t) + ip_y(t) \quad (1.6)$$

Thus by applying the change of variables 1.6 on the x and y components of equation 1.1 and applying the same relation in equation 1.3 with the x and y components of velocity we obtain

$$\frac{d\eta}{dt} = \frac{eB}{c}\left(v_y - iv_x\right) = \frac{eBc}{E_{kin}}\left(p_y - ip_x\right) = \frac{-ieBc}{E_{kin}}\eta \Rightarrow \frac{d\eta}{\eta} = -id\phi \quad (1.7)$$

where

$$d\phi = \frac{eBc}{E_{kin}}dt \Rightarrow \phi = \int \frac{eBc}{\sqrt{\kappa_0^2 + (eEct)^2}}dt = \frac{B}{E}\sinh^{-1}\left(\frac{eEt}{\kappa_0}\right) \quad (1.8)$$

Thus the solution of the equation 1.7 is given by

$$\eta\big(\phi(t)\big) = Ke^{-i\phi} \ (1.9)$$

Since we are interested in the x and y components of the velocity to obtain the position along these axes, note that

$$\eta(t) = p_x(t) + ip_y(t) = \frac{E_{kin}}{c^2}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) = \frac{eB}{c}\left(\frac{dx}{d\phi} + i\frac{dy}{d\phi}\right) \ (1.10)$$

The motion along the x-axis and the y-axis are given by the real and imaginary part of equations 1.9 and 1.10 respectively

$$x\big(\phi(t)\big) = \frac{cK}{eB}\int \cos\phi \, d\phi = \frac{cK}{eB}\sin\phi \ \ (1.11)$$

$$y\big(\phi(t)\big) = \frac{cK}{eB}\int -\sin\phi \, d\phi = \frac{cK}{eB}\cos\phi \ \ (1.12)$$

We can also represent the position along the z-axis as function of $\phi$ by solving for t in equation 1.8 and substituting this expression in equation (1.5) to obtain

$$z\big(\phi(t)\big) = \frac{\kappa_0}{eE}\sqrt{1 + (\sinh(\frac{E\phi}{B}))^2} = \frac{\kappa_0}{eE\phi}\cosh(\frac{E\phi}{B}) \ \ (1.13)$$

From equations 1.11, 1.12 and 1.13 we see that the motion of the particle is a helix. It should be noted that while the transverse momentum quantity

$$p_\perp(t) = \sqrt{p_x(t)^2 + p_y(t)^2} = K \ (1.14)$$

is indeed a constant of motion, the transverse velocity is not

$$v_\perp(t) = \frac{c^2}{E_{kin}}\sqrt{p_x(t)^2 + p_y(t)^2} = \frac{Kc^2}{\sqrt{\kappa_0^2 + (eEct)^2}} \ (1.15)$$

## Problem 1.B Solution

For a magnetic field along the z-axis and an electric field along the y-axis with the same magnitude, the equations of motion are given by

$$\frac{dp_x}{dt} = \frac{eE}{c} v_y, \qquad \frac{dp_y}{dt} = eE\left(1 - \frac{v_x}{c}\right), \qquad \frac{dp_z}{dt} = 0 \ (2.1)$$

By considering the time derivative of the kinetic energy of the particle and the vector form of the equations of motion as seen in equation 1.1, we obtain the following relation

$$\frac{dE_{kin}}{dt} = \frac{d}{dt}(\gamma mc^2) = \vec{v}.\frac{d\vec{p}}{dt} = \vec{v}.\left(e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}\right) = e\vec{v}.\vec{E} = eEv_y \ (2.2)$$

where the last equality comes from the electric field pointing along the y-axis only. By combining the x component of equation 2.1 with equation 2.2 with get the following relation of energy and momentum along x

$$\frac{dE_{kin}}{dt} = c\frac{dp_x}{dt} \Rightarrow \frac{d}{dt}(E_{kin} - cp_x) = 0 \ (2.3)$$

Thus, we see that the z component of equation 2.1 and equation 2.3 are constants with respect to time. By using the definition of the kinetic energy, we can obtain an expression for the x component of momentum as a function of the y component of momentum

$$E_{kin}^2 = (mc^2)^2 + \left(p_x^2 + p_y^2 + p_z^2\right)c^2 \Rightarrow E_{kin}^2 - p_x^2c^2 = p_y^2 + \delta^2 \ (2.4)$$

where the constants terms of mass energy and z component of momentum have been combined into the constant $\delta^2$. By factoring out the last step in equation 2.4 we obtain

$$(E_{kin} - p_xc)(E_{kin} + p_xc) = \alpha(E_{kin} + p_xc) = p_y^2 + \delta^2 \ (2.5)$$

where $\alpha$ is the constant determined by equation 2.3. By solving for the kinetic energy in equation 2.3 and substituting this value in equation 2.6 and solving for x component of momentum we obtain

$$p_x = \frac{-\alpha}{2c} + \frac{(cp_y)^2 + \delta^2}{2\alpha c} \ (2.6)$$

Similarly, by solving for the x component of momentum in equation 2.3 and substituting this value in equation 2.5 we obtain the following expression of the kinetic energy as a function of the y component of momentum

$$E_{kin} = \frac{\alpha}{2} + \frac{(cp_y)^2 + \delta^2}{2\alpha} \ (2.7)$$

If we now multiply equation 2.7 with the y component in equation 2.1 we can obtain an expression of the y momentum component as a function of time

$$E_{kin}\frac{dp_y}{dt} = eE\left(E_{kin} - \frac{E_{kin}v_x}{c}\right) = eE(E_{kin} - cp_x) = \alpha eE \text{ (2.8)}$$

$$\Rightarrow \left(\frac{\alpha}{2} + \frac{(cp_y)^2 + \delta^2}{2\alpha}\right)dp_y = \alpha eE\ dt \ \Rightarrow \left(1 + \left(\frac{\delta}{\alpha}\right)^2\right)p_y + \frac{c^2p_y^3}{3\alpha^2} = 2eEt \text{ (2.9)}$$

To obtain the motion of the particle, we will solve for the coordinates as a function of the y component of momentum by applying the following chain rule which is obtained from equation 2.8

$$\frac{dp_y}{dt} = \frac{eE\alpha}{E_{kin}} \text{ (2.10)}$$

Thus, the coordinates of the particle are given by

$$\frac{dx}{dt} = \frac{dx}{dp_y}\frac{eE\alpha}{E_{kin}} = \frac{c^2p_x}{E_{kin}} = \frac{c^2}{E_{kin}}\left(\frac{-\alpha}{2c} + \frac{(cp_y)^2 + \delta^2}{2\alpha c}\right) \text{(2.11)}$$

$$\Rightarrow x\left(p_y(t)\right) = \frac{c^2}{eE\alpha}\int\left(\frac{-\alpha}{2c} + \frac{(cp_y)^2 + \delta^2}{2\alpha c}\right)dp_y = \frac{c}{2eE}\left(-1 + \left(\frac{\delta}{\alpha}\right)^2\right)p_y + \frac{(cp_y)^3}{6eE\alpha^2} \text{ (2.12)}$$

And similarly for the y and z coordinates, one obtains

$$y\left(p_y(t)\right) = \frac{(cp_y)^2}{2eE\alpha}, \qquad z\left(p_y(t)\right) = \frac{c^2p_yp_z}{eE\alpha} \text{ (2.13)}$$

## Appendix

- Proof that the quantity $(p_x(t)c)^2 + (p_y(t)c)^2$ is a constant of motion for a relativistic charged particle moving through a region with magnetic field perpendicular to the xy-plane.

Starting from the x and y components of equation 1.1, we have that

$$\frac{dp_x}{dt} = \frac{e}{c} v_y B \Rightarrow \gamma \frac{dp_x}{dt} = \frac{dp_x}{d\tau} = \frac{eB}{c} \gamma v_y = \frac{eB}{mc} P_y \ (A.1)$$

$$\frac{dp_y}{dt} = -\frac{e}{c} v_y B \Rightarrow \frac{dp_y}{d\tau} = -\frac{eB}{mc} P_x \ (A.2)$$

where we use the fact that the proper time of the particle is given by $\tau = \frac{t}{\gamma}$. We can decouple this set of differential equations by differentiating with respect to the proper time on equation A.1 to obtain

$$\frac{d^2 p_x}{d\tau^2} = \frac{eB}{mc} \frac{dp_y}{d\tau} = -\left(\frac{eB}{mc}\right)^2 p_x \ (A.3)$$

which has solutions given by

$$p_x(\tau(t)) = c_1 \cos(\frac{eB}{mc}\tau) + c_2 \sin(\frac{eB}{mc}\tau) \ \ (A.4)$$

We can obtain the momentum along the y-axis by direct substitution of equation A.4 in equation A.1, this way one obtains

$$p_y(\tau(t)) = -c_1 \sin\left(\frac{eB}{mc}\tau\right) + c_2 \cos(\frac{eB}{mc}\tau) \ (A.5)$$

From here it is clear that the quantity $(p_x(t)c)^2 + (p_y(t)c)^2$ is a constant of motion.