(a) Given $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A_\mu A^\mu$,

find the equation of fields using Euler-Lagrange equation

Ans: $\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\beta} = \frac{\partial \mathcal{L}}{\partial A_\beta}$

The first term is the field tensor for the classical electromagnetic field with no interaction, therefore the equation for the first term:

$$\partial_\alpha \frac{\partial}{\partial \partial_\alpha A_\beta}\left(-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}\right) = \frac{1}{4\pi} \partial_\mu F^{\mu\beta}$$

As for the second term,

$$\frac{\partial}{\partial A_\beta} \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A_\mu A^\mu = \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 \left(\frac{\partial A_\mu}{\partial A_\beta} A^\mu + \frac{\partial A^\mu}{\partial A_\beta} A_\mu\right)$$

$$= \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 \left(\delta_{\mu\beta} A^\mu + g^{\mu\alpha} \frac{\partial A_\alpha}{\partial A_\beta} A_\mu\right)$$

$$= \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 \left(A^\beta + g^{\mu\alpha} \delta_{\alpha\beta} A_\mu\right)$$

$$= \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 (A^\beta + A^\beta) = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\beta$$

Therefore, equation of the field is:

$$\frac{1}{4\pi} \partial_\mu F^{\mu\beta} + \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\beta = 0, \quad \partial_\mu F^{\mu\beta} + \left(\frac{mc}{\hbar}\right)^2 A^\beta = 0$$

(b) What should m be set equal to in order to recover Maxwell's equation for free space

Ans: $m = 0$.