Chapter 5

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Consider a cube with edge lengths h centered on the origin with faces parallel to the xy, yz, and xz planes with charges q distributed uniformly in the $+x$ half and $-q$ in the $-x$ half.

(a) Find the dipole moments.

(b) Find the potential due to the dipole moments using the multipole expansion

(c) Find the total energy of the charge in an external electric field $\vec{E}_0 = E_0\hat{x}$ (the quadrupoles are all zero for this object).

Part a

Computing the dipoles moments is easy using (5.18)

$$p_i = \int d^3r \rho(\vec{r}) r_i \tag{1}$$

By symmetery, only p_1 will be nonzero

$$p_1 = \int d^3r \rho(x) x = h^3 \frac{q}{h^3} \frac{1}{4} = \frac{q}{4} \tag{2}$$

Part b

The only nonzero moment is

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y) = -\sqrt{\frac{3}{8\pi}}\frac{q}{4}. \tag{3}$$

Now

$$q_{\ell,-m} = (-1)^m q^*_{\ell m} \tag{4}$$

so that $q_{1,-1} = -q_{11}$. Then the potential is given by

$$\Phi(\vec{r}) = -\frac{4\pi}{3}\sqrt{\frac{3}{8\pi}}\frac{q}{4}\frac{Y_{11}(\theta,\phi)}{r^2} + \frac{4\pi}{3}\sqrt{\frac{3}{8\pi}}\frac{q}{4}\frac{Y_{1,-1}(\theta,\phi)}{r^2} \tag{5}$$

$$= \sqrt{\frac{\pi}{24}}\frac{q}{4r^2}\left(\sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta + \sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right) \tag{6}$$

$$= \frac{q}{16r^2}\cos\phi\sin\theta \tag{7}$$

Part c

We can compute the energy of the charge in the electric field using (5.35)

$$U = q\Phi_0 - \vec{p}\cdot\vec{E}_0 - \frac{1}{6}Q_{ij}\partial_i E_{0j}, \tag{8}$$

since q and all Q_{ij} are zero, only the $\vec{p}\cdot\vec{E}_0$ term is non zero. Then

$$U = -\frac{qE_0}{4}. \tag{9}$$