Practice Problem Solution - Chapter 5 - Multipole Expansions

1. A uniformly charged cylinder of radius R, height h, and total charge Q is centered at the origin, with its symmetry axis along the $\hat{z}$ axis and with $-h/2 \leq z \leq h/2$.

a. Obtain the first two non-zero terms in the multi-pole expansion for the electrostatic potential, $\Phi(r,\theta,\phi)$.

b. Obtain the first two non-zero terms in the multi-pole expansion for the electric field, $\mathbf{E}(r,\theta,\phi)$.

The expansion is of the form:

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}}{r^{l+1}}$$

where q_{lm} are the multipole moments of the charge distribution $\rho = Q/(\pi R^2 h)$.

$$q_{lm} = \int r'^l \rho(\overrightarrow{r}') Y^*_{lm}(\theta', \phi') d^3r'$$

The lowest multipole moments are:

$$q_{00} = \frac{1}{\sqrt{4\pi}} Q$$

$$q_{10} = -\frac{3}{4\pi} p_z$$

$$q_{11} = -\frac{3}{8\pi}(p_x - ip_y)$$

$$q_{20} = \sqrt{\frac{5}{16\pi}} Q_{33}$$

$$q_{21} = -\sqrt{\frac{15}{72\pi}}(Q_{13} - iQ_{23})$$

$$q_{22} = \sqrt{\frac{15}{288\pi}}(Q_{11} - 2iQ_{12} - Q_{22})$$

where

$$p_i = \int \rho(\overrightarrow{r}) d^3r$$

$$Q_{ij} = \int \rho(\overrightarrow{r})(3r_ir_j - r^2\delta_{ij}) d^3r$$

$$p_x = p_y = p_z = 0 \Rightarrow q_{10} = q_{11} = 0$$

For any axially symmetric distribution $\rho(\overrightarrow{r})$ the quadrupole moment tensor has the form:

$$Q_{ij} = \begin{Bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -2A \end{Bmatrix}$$

Performing the integrals:

$$Q_{xx} = Q_{yy} = \frac{1}{4}QR^2 - \frac{1}{12}Qh^2 \Rightarrow q_{22} = 0$$

$$Q_{zz} = -\frac{1}{2}QR^2 + \frac{1}{6}Qh^2$$

Therefore:

$$\Phi(\overrightarrow{r}) = \frac{Q}{r} + \frac{1}{4r^3}Q_{zz}(3\cos^2\theta - 1)$$

For part (b) remember that

$$E(r, \theta\,\phi) = -\bigtriangledown\Phi(r, \theta, \phi)$$

$$E_r = \frac{Q}{r^2} + \frac{3}{4r^4}Q_{zz}(3\cos^2\theta - 1)$$

$$E_\theta = \frac{3}{2r^4}Q_{zz}\cos\theta\sin\theta$$

$$E_\phi = 0$$