Chapter 6 review solution

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1 Part a

The magnetic moment is given by

$$\vec{m} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J} \tag{1}$$

To find $\vec{J}$:

$$\vec{J} = \rho\vec{v} \tag{2}$$

$$\rho = \sigma\delta(r-a) \tag{3}$$

$$\vec{v} = \vec{\omega}\times\vec{r} = \omega a\sin(\theta)\hat{\phi} \tag{4}$$

Therefore, $\vec{J}$ is

$$\vec{J} = \omega a\sigma\sin(\theta)\delta(r-a)\hat{\phi} \tag{5}$$

Now we can determine $\vec{m}$.
First we calculate the cross product of $\vec{r}$ and $\vec{J}$.

$$\vec{r}\times\vec{J} = -r\omega a\sigma\sin(\theta)\delta(r-a)\hat{\theta} \tag{6}$$

Now we do the integral given in Equation (1).

$$\vec{m} = -\frac{1}{2}\omega a\sigma\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r^3\sin^2(\theta)\delta(r-a)d\theta d\phi dr\ \hat{\theta} \tag{7}$$

To do this integral, we must express the unit vector $\hat{\theta}$ as:

$$\hat{\theta} = \hat{\rho}\sin\theta - \hat{z}\cos\theta \tag{8}$$

The $\hat{\rho}$ term will integrate to zero, so the integral will become

$$\vec{m} = -\frac{1}{2}\omega a\sigma\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^3\sin^3(\theta)\delta(r-a)drd\theta d\phi\ \hat{z} \tag{9}$$

$$\boxed{\vec{m} = \frac{4}{3}\omega a^4\sigma\pi\ \hat{z}} \tag{10}$$

2 Part b

The magnetic field at a point $\vec{r}$ far away from the magnetic moment is given by

$$\vec{B}(\vec{r}) = -\frac{\vec{m}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{m} \cdot \vec{r}) \tag{11}$$

Plugging in our expression for the magnetic moment (Equation (10)), we find:

$$\vec{B}(\vec{r}) = -\frac{4\sigma\omega\pi a^4}{3d^3}\hat{z} + \frac{3d\hat{r}}{d^5}\left(\frac{4}{3}\sigma\omega\pi a^4\hat{z} \cdot d\hat{z}\right) \tag{12}$$

$$\vec{B}(\vec{r}) = -\frac{4\sigma\omega\pi a^4}{3d^3}\,\hat{z} + \frac{4\sigma\omega\pi a^4}{d^3}\,\hat{z} \tag{13}$$

Simplifying,

$$\boxed{\vec{B}(\vec{r}) = \frac{8\sigma\omega\pi a^4}{3d^3}\,\hat{z}} \tag{14}$$