Electrodynamics Subject Exam Prep Quiz, Friday, April 29, 2017

your name(s)____

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

E & M Study Guide

Ch 1 Special Relativity Primer

1.1 Gamma Factors and Such

- $t_{lab} = \gamma t_0$
- $L_{lab} = \frac{L_0}{\gamma}$

1.2 Lorentz Transformations

- $r^\alpha = L^\alpha_\beta r'^\beta$
- $$L^\alpha_\beta = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ for } \hat{x} \text{ boost } \gamma^2(1-v^2) = 1$$
- $sinh\eta \equiv \gamma v$, $cosh\eta \equiv \gamma$, $\eta =$ "rapidity"

1.3 invariants and the Metric Tensor $g^{\alpha\beta}$

- Dot product: $A^\alpha g_{\alpha\beta} B^\beta = A^\alpha B_\alpha =$ invariant
- $$g_{\alpha\beta} = g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
- $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$
- Continuity Eqn: $\partial \cdot J = 0$
- Maxwell's Eqns:
  $\partial_\alpha F^{\alpha\beta} = J^\beta$
  $\partial_\alpha \tilde{F}^{\alpha\beta} = 0$

1.4 Four-Velocities and Momenta

- $u \equiv (\gamma, \gamma \frac{d\vec{x}}{dt}$
- $u^\alpha u_\alpha = 1$
- $p \equiv mu$
- $p^\alpha p_\alpha = m^2$

Ch 2 Dynamics of a Relativistic Point

2.1 Lagrangian for a Free Relativistic Particle

- $S = -m \int dt \sqrt{(\frac{dt'}{dt})^2 - (\frac{d\vec{r}'}{dt})^2}$, $r'^\alpha = r^\alpha + \delta\Omega^{\alpha\beta}(t) r_\beta$
- $\delta S = \int dt \frac{\partial \mathcal{L}}{\partial r^\alpha} \frac{d}{dt}(\delta\Omega^{\alpha\beta} r_\beta)$
  $= \int dt[-\frac{d}{dt}(\pi_\alpha r_\beta) + \pi_\alpha \dot{r}_\beta] \delta\Omega^{\alpha\beta}$
  $= 0$
- $\frac{d}{dt}(r^\alpha p^\beta - p^\alpha r^\beta) = 0$

2.2 Interaction of a Charged Particle with an External EM Field

- $S = \int d\tau(-m - eu \cdot A)$
  $= -m \int dt \sqrt{(\frac{dt'}{dt})^2 - (\frac{d\vec{r}'}{dt})^2} - e \int dt (A_0 - \vec{v} \cdot \vec{A})$
- $\mathcal{L} = -m\sqrt{1-v^2} - e((A_0 - \vec{v} \cdot \vec{A})$
- $\frac{d}{dt}(\gamma m v_i) = -e\partial_i A_0 - e\partial_t A_i + e\vec{v} \times (\vec{\Delta} \times \vec{A})$
- $\vec{E} = -\vec{\Delta} A_0 - \partial_t \vec{A}$
- $\vec{B} = \vec{\Delta} \times \vec{A}$
- $\frac{d}{dt}(\gamma m \vec{v}) = e\vec{E} + e\vec{v} \times \vec{B}$

2.3 Motion in a Constant Magentic Field

- $\vec{A} = xB\hat{y}$, $\vec{B} = B\hat{z}$
- $S = -\int dt[m\sqrt{1-v^2} - exBv_y]$
- $\frac{d}{dt}(\gamma m \vec{v}) = e(v_y \hat{x} - v_x \hat{y})B$
- $\dot{v}_x = \frac{eB}{\gamma m} v_y = \omega v_y$
- $\dot{v}_y = -\frac{eB}{\gamma m} v_x = -\omega v_x$
- $v_x = \omega R sin(\omega t)$, $x = Rcos(\omega t)$
- $v_y = -\omega R cos(\omega t)$, $y = Rsin(\omega t)$

Gauge Transformations

- $A'^\mu = A^\mu + \partial^\mu \Lambda(t, x, y, z)$
- $\vec{E}, \vec{B}$ do not change

2.5 The EM Field Tensor

- $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F^\alpha_\beta \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

- $\frac{d}{d\tau} u^\alpha = eF^{\alpha\beta} u_\beta$

Ch 3 Dynamics of EM Fields

3.1 Lagrangian (Density) for Free Fields: Deriving Maxwell's Equations

- $S_f = \int d^4 r \mathcal{L}(a^\mu, \partial_\nu A^\mu)$
- $\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} = \frac{\partial \mathcal{L}}{\partial A^\nu}$
- $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{1}{16\pi} F^{\mu\nu} F_{\nu\mu}$
- $S_m = \int d^4 r J \cdot A$
- $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$
- $J^\alpha \equiv \frac{1}{\Omega} \sum_{a \in \Omega} q_a \frac{u^\alpha_a}{\gamma_a}$, Ω is 4D volume
- $\vec{\Delta} \cdot \vec{E} = 4\pi J^0$
- $(\vec{\Delta} \times \vec{B}) - \partial_t \vec{E} = 4\pi \vec{J}$
- Dual EM tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

- $\partial_\mu \tilde{F}^{\mu\nu} = 0$
- $\vec{\Delta} \cdot \vec{B} = 0$
- $\partial_t \vec{B} + \vec{\Delta} \times \vec{E} = 0$

3.2 Pseudo-Vectors and Pseudo-Scalers

- $F^{\alpha\beta} \tilde{F}_{\alpha\beta} = -4\vec{E} \cdot \vec{B}$, pseudoscalar
- $F^{\alpha\beta} F_{\alpha\beta} = -2(|\vec{E}|^2 - |\vec{B}|^2)$, (regular) scalar

The Stress-Energy Tensor of the EM Field

- $T^{\alpha\beta} \equiv pi^\alpha \partial^\beta \phi - g^{\alpha\gamma} \mathcal{L}$, $\phi = A^\mu$, $\pi^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}$
- $T^{\alpha\gamma} = \frac{1}{4\pi} F^{\alpha\beta} F_\beta{}^\gamma + F^{\mu\nu} F_{\mu\nu}$
- $T^{00} = \frac{1}{8\pi}(E^2 + B^2)$
- $T^{0i} = \frac{1}{4\pi} \epsilon_{ijk}(E_j B_k)$
- $T^{ij} = \frac{1}{8\pi}(\delta_{ij}(E^2 + B^2) - 2E_i E_j - 2B_i B_j)$
- $U^{(f)} = \frac{1}{2} \int d^3 r A_0(\vec{r}) J_0(\vec{r})$

3.4 Hyper-Surfaces and Conservation of $E, \vec{P}, \vec{L}$

- stress-energy tensor is symmetric

Electrostatics

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\vec{E} = -\nabla\Phi$$
$$\nabla^2\Phi = -4\pi\rho$$
$$PE = \frac{1}{2}\int d^3r\rho(\vec{r})\Phi(\vec{r})$$
$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$$

Laplace Equation: $\nabla^2\Phi = 0$
Laplace Operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$
$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$

Solution, Cylindrical Coordinates:

The large and small x expansions are:

$$x \ll 1, \qquad J_m(x) \rightarrow \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m, \tag{4.43}$$
$$N_m(x) \rightarrow \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & m = 0 \\ -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m, & m \neq 0. \end{cases},$$

and the expansion for large x are:

$$x \gg 1, m \qquad J_m(x) \rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right), \tag{4.44}$$
$$N_m(x) \rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right).$$

Each of these expressions assumes $m \geq 0$, and the constant 0.57772 is Euler's constant.

Solution, Spherical Coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l,m}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]Y_l^m(\theta,\phi)$$
$$\Phi(r,\theta,\phi) = \sum_{l}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

Orthogonality relations:

$$\int (Y_l^m)^* \, Y_{l'}^{m'} \; d\Omega = \delta_{ll'}\delta_{mm'}$$

$$\int_{-1}^{1} P_l(x) P_{l'}(x) \; dx = \delta_{ll'} \frac{2}{2l+1}$$

Spherical harmonics of order ≤ 2:

$$Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{-i\phi} \sin\theta$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$$

$$Y_1^1 = \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{i\phi} \sin\theta$$

$$Y_2^{-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{-2i\phi} \sin^2\theta$$

$$Y_2^{-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}} e^{-i\phi} \sin\theta\cos\theta$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1)$$

$$Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} e^{i\phi} \sin\theta\cos\theta$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{2i\phi} \sin^2\theta$$

Legendre polynomials of order ≤ 3:

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3 = \tfrac{1}{2}(5x^3 - 3x)$$

Multipole Expansions

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{Q}{r}$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\Phi(\vec{r}) = \sum_{lm} \frac{4\pi}{(2l+1)} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

Multipole Moments

$$q_{00} = \sqrt{\frac{1}{4\pi}} q$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

$$q_{10} = -\sqrt{\frac{3}{4\pi}} p_z$$

$$q_{22} = \sqrt{\frac{15}{288\pi}} (Q_{11} - 2iQ_{12} - Q_{22})$$

$$q_{21} = -\sqrt{\frac{15}{72\pi}} (Q_{13} - iQ_{23})$$

$$q_{20} = \sqrt{\frac{5}{16\pi}} Q_{33}$$

Monopole Moment

$$q = \int d^3r \rho(\vec{r}) \tag{1}$$

Dipole moments

$$p_i = \int d^3r \rho(\vec{r}) r_i \tag{2}$$

Quadrupole moments

$$Q_{ij} = \int d^3r \rho(\vec{r})(3r_i r_j - r^2 \delta_{ij}) \tag{3}$$

Magnetostatics

Vector Potential

$$\vec{A}(\vec{r}) = \int d^3r' \, \vec{J}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \tag{4}$$

Biot-Savart Law

$$\vec{B}(\vec{r}) = \int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \tag{5}$$

Net force on a current density $\vec{J}$ due to a magnetic field

$$\vec{F} = \int d^3r \vec{J} \times \vec{B} \tag{6}$$

Magnetic Moment

$$\vec{m} = \frac{1}{2} \int d^3r \, \vec{r} \times \vec{J} \tag{7}$$

Magnetic field due to magnetic moment

$$\vec{B} = -\frac{\vec{m}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{m} \cdot \vec{r}) \tag{8}$$

Torque due to magnetic field

$$\vec{\tau} = \vec{m} \times \vec{B} \tag{9}$$

Potential energy of magnetic moment in a magnetic field

$$U = -\vec{m} \cdot \vec{B} \tag{10}$$

For intrinsic spin $\vec{S}$, the magnetic moment is

$$\vec{\mu} = g \frac{e\vec{S}}{2m} = g \frac{\vec{S}}{\hbar} (\frac{e\hbar}{2m}) \tag{11}$$

Useful g factors:

electron	-2.002319
muon	-2.002332
proton	5.585695
neutron	-3.826085

your name(s)____

LONG ANSWER SECTION

1. Consider a thin spherical shell of radius $\boldsymbol{R}$ with charge density

$$\boldsymbol{\sigma = \sigma_0 \cos\theta},$$

where $\boldsymbol{\theta}$ is the polar angle measured relative to the $\boldsymbol{z}$ axis.

(a) (5 pts) State the conditions relating the electric potential $\boldsymbol{\Phi}$ on opposite sides of the shell, $\boldsymbol{R-\epsilon}$ and $\boldsymbol{R+\epsilon}$.

(b) (10 pts) Solve for the potential for both $\boldsymbol{r<R}$ and $\boldsymbol{r>R}$.

(c) (5 pts) What is the electric dipole moment, $\vec{\boldsymbol{d}}$?

a) $$-\left.\frac{\partial \Phi}{\partial r}\right|_{R+\varepsilon} - \left.\frac{\partial \Phi}{\partial r}\right|_{R-\varepsilon} = 4\pi\sigma_0\cos\theta$$

b) $$\Phi = \begin{cases} A\cos\theta\cdot r, & r<R \\ B\cos\theta/r^2, & r>0 \end{cases}$$

$$4\pi\sigma_0 = 2B/R^3 + A$$

$$AR = B/R^2$$

$$A = 4\pi\sigma_0/3$$

$$B = 4\pi\sigma_0 R^3/3$$

$$\Phi = \begin{cases} \frac{4\pi\sigma_0}{3} r, & r<R \\ \frac{4\pi\sigma_0 R^3}{3r^2}, & r>R \end{cases}$$

c) $d_x = d_y = 0$,

$$d_z = 2\pi\int R^2 d\cos\theta\ \sigma_0\cos\theta\cdot R\cos\theta$$

$$= 2\pi R^3\sigma_0\frac{2}{3} = \frac{4\pi\sigma_0 R^3}{3}$$

Extra workspace for #1

your name(s)____

2. A square loop of dimensions $L \times L$ is centered at the origin with its plane pointing along the z axis. Two charges, each $+Q$, move at a fixed speed $v << c$, counter-clockwise around the loop as viewed looking down on the $x-y$ plane from the $+z$ direction. The charges always are positioned on opposite sides of the loop.

   (a) (5 pts) What is the time-averaged magnetic dipole moment?

   (b) (10 pts) What is the magnetic field at a position $(x = 0, y = 0, z)$ as a function of time.

a) $I = 2Q/t$ $\quad\quad vt = 4L$, $\; t = 4L/v$

$= \dfrac{Qv}{2L}$

$m_z = I \cdot A = \dfrac{QvL}{2}$

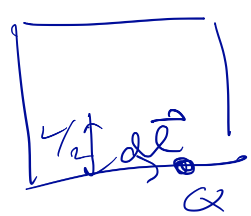
b) At time t, $R = \sqrt{\left(\frac{L}{2}\right)^2 + v^2t^2 + z^2}$

$-\dfrac{L}{2v} < t < L/2v$

$\hat{z} B_z = -\dfrac{\vec{R} \times I\,\vec{dl}}{R^3}$ $\quad\quad \int I\,\vec{dl} = Q\vec{v}$ ← one charge in dl

$\hat{z} B_z = -\dfrac{-\frac{L}{2}\hat{y} \times vQ\hat{L}}{R^3}$ $\quad$ for charge on 1st portion

$\hat{L} = \hat{x}$



$B_z = \dfrac{LvQ/2}{\left[\left(\frac{L}{2}\right)^2 + v^2t^2 + z^2\right]^{3/2}}$

multiply by 2 for other charge

$\vec{B} = \hat{z}\dfrac{vLQ}{\left[\left(\frac{L}{2}\right)^2 + v^2t^2 + z^2\right]^{3/2}}$, $\quad -\dfrac{L}{2v} < t < \dfrac{L}{2v}$

then repeat after $\Delta t = \dfrac{L}{v}$

Extra workspace for #2

your name(s)______________________________________

3. (10 pts) Consider a finite volume with some constant current density $\vec{J}(\vec{r})$. Using current conservation, $\nabla \cdot \vec{J} = 0$, prove that

$$\int d^3r \; \vec{J}(\vec{r}) = 0.$$

$$\int d^3r \, \vec{r}(\nabla \cdot \vec{J}) = 0 =$$

$$\int d^3r \; r_i \; \partial_k J_k = 0$$

$$-\int d^3r \; J_k (\partial_k r_i) = 0$$

$$-\int d^3r \; J_k \, \delta_{ki} = 0 = \int d^3r \; J_i$$

your name(s)____

Extra work space for #3

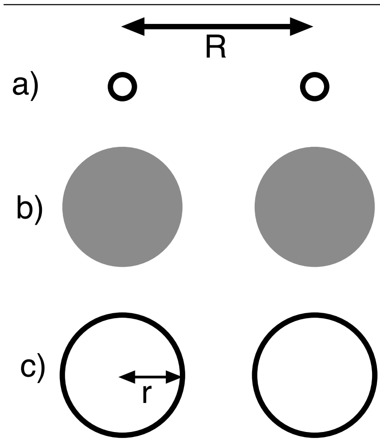
your name(s)______________________________

4. YES or NO (4 pts)

   (a) A configuration of charges has no electric monopole or dipole moments, but does have an electric quadrupole moment. Will the quadrupole field as seen by an observer far away be changed by moving the entire distribution by a finite displacement $\vec{a}$?
   NO

   (b) A configuration of charges has no electric monopole but does have a non-zero electric dipole moment and a non-zero electric quadrupole moment. Will the dipole field as seen by an observer far away be changed by moving the entire distribution by a finite displacement $\vec{a}$?
   NO

   (c) A configuration of charges has no electric monopole but does have a non-zero electric dipole moment and a non-zero electric quadrupole moment. Will the quadrupole field as seen by an observer far away be changed by moving the entire distribution by a finite displacement $\vec{a}$?
   YES



5. (4 pts) Consider the three charge configurations shown above:
(*a*) two point charges $+Q$ separated by R
(*b*) two spheres of radius $r < R/2$, each with charge $+Q$ uniformly spread throughout the volume, with the centers separated by R
(*c*) two spherical shells of radius $r < R/2$, each with charge $+Q$ uniformly spread throughout the surface, with the centers separated by R.

   The work required to move the spheres (or points) from infinity to the separation R is labeled W_a, W_b and W_c for each configuration. Label each statement as true or false.

   (a) $W_a > W_b$ false
   (b) $W_a > W_c$ false
   (c) $W_b > W_c$ false