Electrodynamics Subject Exam Prep Quiz, Tuesday, May 2, 2017

your name____________________________

$$\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) &= 0,\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\vec{a}) &= 0,\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{a}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\boldsymbol{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\boldsymbol{\nabla}\psi+\psi\boldsymbol{\nabla}\cdot\vec{a},\\
\boldsymbol{\nabla}\times(\psi\vec{a}) &= \boldsymbol{\nabla}\psi\times\vec{a}+\psi\boldsymbol{\nabla}\times\vec{a},\\
\boldsymbol{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\boldsymbol{\nabla})\vec{b}+(\vec{b}\cdot\boldsymbol{\nabla})\vec{a}+\vec{a}\times(\boldsymbol{\nabla}\times\vec{b})+\vec{b}\times(\boldsymbol{\nabla}\times\vec{a}),\\
\boldsymbol{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\boldsymbol{\nabla}\times\vec{a})-\vec{a}\cdot(\boldsymbol{\nabla}\times\vec{b}),\\
\boldsymbol{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\boldsymbol{\nabla}\times\vec{b})-\vec{b}(\boldsymbol{\nabla}\times\vec{a})+(\vec{b}\cdot\boldsymbol{\nabla})\vec{a}-(\vec{a}\cdot\boldsymbol{\nabla})\vec{b},\\
\boldsymbol{\nabla}\cdot\vec{r} &= 3,\\
\boldsymbol{\nabla}\times\vec{r} &= 0,\\
\boldsymbol{\nabla}\cdot\hat{r} &= 2/r,\\
\boldsymbol{\nabla}\times\hat{r} &= 0,\\
(\vec{a}\cdot\nabla)\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r}.
\end{aligned}$$

$$\begin{aligned}
\int_V d^3r\ \boldsymbol{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \boldsymbol{\nabla}\psi &= \int_S \psi d\vec{S},\\
\int_V d^3r\ \boldsymbol{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi) &= \int_S \phi(d\vec{S})\cdot\boldsymbol{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\boldsymbol{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S (d\vec{S})\times\boldsymbol{\nabla}\psi &= \oint_C d\vec{\ell}\,\psi.
\end{aligned}$$

$$\begin{aligned}
\nabla^2 &= \partial_r^2+\frac{2}{r}\partial_r-\frac{\ell(\ell+1)}{r^2},\\
\nabla^2 &= \partial_\rho^2+\frac{1}{\rho}\partial_\rho-\frac{m^2}{r^2},\\
\nabla^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}).
\end{aligned}$$

$$L^{\alpha}{}_{\beta} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

$$\begin{aligned} m\frac{d}{d\tau}u^{\alpha} &= eF^{\alpha\beta}u_{\beta}, \\ \frac{d\vec{p}}{dt} &= e\vec{E} + e\vec{v}\times\vec{B}, \\ \omega_c &= \frac{eB}{\gamma m}. \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\vec{E} &= 4\pi J^0, \\ (\boldsymbol{\nabla}\times\vec{B}) - \partial_t\vec{E} &= 4\pi\vec{J}, \\ \boldsymbol{\nabla}\cdot\vec{B} &= 0, \\ \partial_t\vec{B} + \boldsymbol{\nabla}\times\vec{E} &= 0, \\ \partial_\alpha F^{\alpha\beta} &= 4\pi J^{\beta}, \\ \partial_\alpha \tilde{F}^{\alpha\beta} &= 0. \end{aligned}$$

$$e^2 = \frac{\hbar c}{137.036},$$

$$\begin{aligned} T^{\alpha\beta} &= \pi^{\alpha}\partial^{\beta}\phi - g^{\alpha\beta}\mathcal{L}, \\ \pi^{\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} \end{aligned}$$

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(E^2+B^2), \\ T^{0i} &= \frac{1}{4\pi}\epsilon_{ijk}E_jB_k, \\ T^{ij} = -T^{i}{}_{j} &= \frac{1}{8\pi}(\delta_{ij}(E^2+B^2) - 2E_iE_j - 2B_iB_j). \end{aligned}$$

$$\begin{aligned} \vec{E} &= -\boldsymbol{\nabla}A_0 - \partial_t\vec{A} = -\boldsymbol{\nabla}\Phi - \partial_t\vec{A}, \quad \vec{B} = \boldsymbol{\nabla}\times\vec{A}, \\ P_\ell(\cos\theta) &= \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell m=0}(\theta), \\ Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \\ Y_{1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi}, \qquad Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \\ Y_{2,\pm1} &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}, \qquad Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \\ Y_{\ell-m}(\theta,\phi) &= (-1)^m Y^{*}_{\ell m}(\theta,\phi), \\ \delta_{\ell\ell'}\delta_{mm'} &= \int d\Omega\; Y_{\ell,m}(\theta,\phi)Y_{\ell',m'}(\theta,\phi), \\ P_0(x) &= 1, \qquad P_1(x) = x, \\ P_2(x) &= \frac{1}{2}(3x^2-1), \qquad P_3(x) = \frac{1}{2}(5x^3-3x), \\ P_\ell(x=1) &= 1, \qquad \int_{-1}^{1}dx\; P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}, \\ (3D)\quad \Phi &= \sum_{\ell m}\left(A_{\ell m}r^{\ell} + B_{\ell m}r^{-\ell-1}\right)Y_{\ell m}(\theta,\phi)e^{im\phi}, \\ (2D)\quad \Phi &= A_0\ln(\rho) + \sum_m e^{im\phi}\left(A_m r^m + B_m r^{-m}\right), \\ \Phi &= A_0J_0 = \sum_m e^{im\phi}\left(A_mJ_m(k\rho) + B_mN_m(k\rho)\right)e^{\pm kz}, \\ \Phi &= \frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \sum_m \frac{4\pi}{5r^3}q_{2m}(r)Y_{2m}(\theta,\phi), \\ \vec{E} &= -\frac{1}{r^3}\vec{p} + 3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} + \cdots, \\ \Phi(r,\theta,\phi) &= \sum_{\ell m}\frac{4\pi}{(2\ell+1)r^{\ell+1}}q_{\ell m}(r)Y_{\ell m}(\theta,\phi), \\ q_{22} &= \sqrt{\frac{15}{32\pi}}\int d^3r\; \rho(\vec{r})(x-iy)^2 = \sqrt{\frac{15}{288\pi}}(Q_{11} - 2iQ_{12} - Q_{22}), \\ q_{21} &= -\sqrt{\frac{15}{8\pi}}\int d^3r\; \rho(\vec{r})(x-iy)^2 = -\sqrt{\frac{15}{72\pi}}(Q_{13} - iQ_{23}), \\ q_{20} &= \sqrt{\frac{5}{16\pi}}\int d^3r\; \rho(\vec{r})(3z^2-r^2) = \sqrt{\frac{5}{16\pi}}Q_{33}, \\ Q_{ij} &\equiv \int d^3r\; (3r_ir_j - r^2\delta_{ij})\rho(\vec{r}), \\ U &= q\Phi_0 - \vec{p}\cdot\vec{E} - \frac{1}{6}Q_{ij}\partial_iE_j, \end{aligned}$$

$$
\begin{aligned}
\nabla^2 A^\alpha &= -4\pi J^\alpha, \\
\vec{m} &= \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J} = \frac{I}{2}\int\ \vec{r}\times d\vec{\ell}, \\
\vec{B} &= -\frac{\vec{m}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{m}\cdot\vec{r}), \\
\mu_e &= g_e\frac{e\hbar}{2m_e}, \\
U &= \frac{(\vec{\mu}_N\cdot\vec{\mu}_e)}{r^3} - \frac{3(\vec{\mu}_N\cdot\vec{r})(\vec{\mu}_e\cdot\vec{r})}{r^5} - \frac{8\pi}{3}(\vec{\mu}_N\cdot\vec{\mu}_e)\delta^3(\vec{r}) \\
&\quad -e\frac{(\vec{\mu}_N\cdot\vec{L})}{mr^3}, \\
T_{00} &= \frac{1}{8\pi}\left(|\vec{E}|^2+|\vec{B}|^2\right) \\
&= \frac{a_i^2+b_i^2}{8\pi} = \frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t), \\
T_{0i} &= \epsilon_{ijk}\frac{E_jB_k}{4\pi} \\
&= \hat{k}_i\frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t), \\
T^{ij} = -T^i{}_j &= \frac{1}{8\pi}(\delta_{ij}(E^2+B^2)-2E_iE_j-2B_iB_j) \\
&= \frac{1}{4\pi}\left\{|\vec{a}|^2\delta_{ij}-a_ia_j-b_ib_j\right\}\cos^2(\vec{k}\cdot\vec{r}-\omega t)
\end{aligned}
$$

$$
\omega_s = \omega\sqrt{\frac{1-v}{1+v}}
$$

$$
\begin{aligned}
(TM)\quad E_z &= \psi(x,y)e^{-i\omega t+ik_zz}, \\
\Psi &= 0 \quad \text{at boundary}, \\
\vec{E}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_zz}\boldsymbol{\nabla}_t\psi(x,y), \\
\vec{B}_t(x,y) &= \left(\frac{\omega}{k_z}\right)\hat{z}\times\vec{E}_t, \\
(TE)\quad B_z &= \psi(x,y)e^{-i\omega t+ik_zz}, \\
(\hat{n}\cdot\boldsymbol{\nabla}_t)\psi(x,y)|_S &= 0, \\
\vec{B}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_zz}\boldsymbol{\nabla}_t\psi(x,y), \\
\vec{E}_t(x,y) &= -\left(\frac{\omega}{k_z}\right)\hat{z}\times\vec{B}_t.
\end{aligned}
$$

$$
\begin{aligned}
A^\alpha(x) &= \int d^4x'\ \frac{1}{|\vec{x}-\vec{x}'|}J^\alpha(x')\delta(x_0-x_0'-|\vec{x}-\vec{x}'|), \\
\vec{E} &= e\left\{\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3|\vec{x}|}\right\}, \\
\vec{B} &= \hat{n}\times\vec{E}.
\end{aligned}
$$

$$
\begin{aligned}
P &= \frac{2e^2}{3c}|\dot{\vec{\beta}}|^2 \quad \text{(Non.Rel.)}, \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\vec{\beta}\cdot\hat{n})^6}|(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2, \\
P &= \frac{2}{3c}e^2\gamma^6\left[\dot{\beta}^2-|\vec{\beta}\times\dot{\vec{\beta}}|^2\right], \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\beta\cos\theta)^5}|\dot{\vec{\beta}}|^2\sin^2\theta \quad \text{(linear)}, \\
P &= \frac{2e^2\dot{\beta}^2}{3c}\gamma^6 \quad \text{(linear)}, \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\beta n_\beta)^5}|\dot{\vec{\beta}}|^2\left((1-\beta n_\beta)^2-(1-\beta^2)n_r^2\right) \quad \text{(circular)}, \\
P &= \frac{2}{3c}e^2\dot{\beta}^2\gamma^4 \quad \text{(circular)}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{8\pi}k^2|\hat{n}\times\vec{p}|^2, \\
P &= \frac{\omega^4}{3}|\vec{p}|^2,
\end{aligned}
$$

$$
\begin{aligned}
\text{(Thomson)}\quad \sigma &= \frac{8\pi e^4}{3m^2}, \\
\frac{\Delta\lambda}{\lambda} &= \frac{\hbar\omega}{m}(1-\cos\theta_s).
\end{aligned}
$$

electron	$-2.00231930436182 \pm 0.00000000000052$
muon	$-2.0023318418 \pm 0.0000000013$
proton	$5.585694702 \pm 0.000000017$
neutron	$-3.82608545 \pm 0.00000090$

your name____________________________

LONG ANSWER SECTION

1. A thin wire extends from $x, y, z = (0, 0, -a)$ to $(0, 0, a)$. The current on the wire is wave-like

$$I = I_0 \cos(kz) \cos \omega_0 t, \quad k = \pi/2a.$$

   - (a) (5 pts) Find the charge density on the wire as a function of time
   - (b) (5 pts) What is the average power radiated by the wire
   - (c) (5 pts) What is the angular distribution of the radiated power?

a) $$\frac{\partial \lambda}{\partial t} = -\partial_z I(z,t) = k I_0 \sin kz \cos \omega_0 t$$

$$\lambda = \frac{k I_0}{\omega_0} \sin kz \sin \omega_0 t$$

b) $$P = \frac{\omega^4}{3} p_z^2 ,$$

$$p_z = \int z \, \lambda(z) \, dz$$

$$= \frac{k I_0}{\omega_0} \int_{-a}^{a} dz \sin kz \cdot z$$

$$= \frac{2 I_0}{\omega_0 k} = \frac{4 a I_0}{\omega_0 \pi}$$

$$\text{Power} = \frac{\omega_0^2}{3} \frac{a^2 I_0^2}{\pi^2} \cdot 16$$

c) $$\frac{dP}{d\Omega} = \frac{1}{8\pi} \omega^4 (\hat{n} \times \vec{p})^2$$

$$= \frac{\omega^2}{8\pi} \frac{16 a^2 I_0^2}{\pi^2} \sin^2 \theta$$

$$\boxed{\times \frac{1}{c^3}}$$

↑ for dimensions

your name____________________________

Extra workspace for #1

your name____________________________

2. A square wave guide has transverse dimensions $0 < x < a$, $0 < y < a$. For a transverse magnetic wave (TM),

   (a) (10 pts) For a wave that propagates in the $+z$ direction with frequency ω, find a solution for the electric and magnetic fields, where the maximum electric field strength is E_0. Choose the solution with the fewest transverse nodes.

   (b) (5 pts) What is the lowest value of ω that leads to a propagating solution? And what is longitudinal velocity of the wave in that limit?

a) $$E_z = E_0\, e^{-i\omega t + ik_z z} \sin\frac{\pi x}{2a} \sin\frac{\pi y}{2a}$$

$$E_x = \frac{ik_z}{\omega^2 - k_z^2}\, \partial_x \left(\sin\frac{\pi x}{a} \sin\frac{\pi y}{2a}\right) e^{-i\omega t + ik_z z} E_0$$

$$= \frac{ik_z}{\omega^2 - k_z^2} \frac{\pi}{a} \cos\frac{\pi x}{a} \sin\frac{\pi x}{a}\, e^{-i\omega t + ik_z z} E_0$$

$$E_y = \frac{ik_z}{\omega^2 - k_z^2} \frac{\pi}{a} \sin\frac{\pi x}{a} \cos\frac{\pi x}{a}\, e^{-i\omega t + ik_z z} E_0$$

$$\omega^2 = k_z^2 + 2\left(\frac{\pi}{a}\right)^2$$

b) $$\text{lowest}\ \omega = \left(\frac{\pi}{a}\right)\sqrt{2}$$

your name__________________________

SHORT ANSWER SECTION

3. A linear accelerator accelerates either electrons or protons with an electric field of strength E_0 along the z direction. In terms of the ratio of masses M_p/M_e,

   (a) (4 pts) Find the ratio of radiative powers, P_p/P_e, at the beginning of the acceleration when both particles would be moving non-relativistically.

   $\boxed{(M_e/m_p)^2}$

   (b) (4 pts) Find the ratio of radiative powers, P_p/P_e, at the end of the acceleration when both particles would be moving ultra-relativistically with energy K.

   $\boxed{(M_p/M_e)^2}$

$\dot{\gamma} \simeq \gamma^3 \dot{\beta}, \quad m\dot{\gamma} \sim m\dot{\beta}\gamma^3 = eE$

$\dot{\beta}^2 \sim \frac{1}{m^2\gamma^6}, \quad P_p/P_e \sim \left(\frac{m_e}{M_p}\right)^2$

4. (4 pts) Two gases of fully ionized particles are stored in identical containers with identical numbers of ions (and electrons). In gas A, the gas is fully ionized hydrogen, where all the positive ions are protons. In gas B, the positive ions are deuterons (one neutron and one proton), which have approximately twice the mass of a proton. The number of free electrons in each container is thus equal. The mean free paths, ℓ_A and ℓ_B, of light traveling through the containers vary by approximately what factor? Circle the correct answer:

   $\ell_A/\ell_B \approx$

   (a) $\frac{1}{16}$
   (b) $\frac{1}{8}$
   (c) $\frac{1}{4}$
   (d) $\frac{1}{2}$
   [✓ (e) 1]
   (f) 2
   (g) 4
   (h) 8
   (i) 16

scattering
dominated
by electrons!