your name(s)____________________________

Physics 841 Quiz #4 - Friday, Feb. 17 and Monday, Feb. 20

You may work in groups of 4 (no more than one person from previous group)

Open note, open book, open internet, open mouth...

1. Consider a sphere of radius R centered at the origin, The surface of the potential is $V(\cos\theta)$.

   (a) In spherical coordinates, using the azimuthal symmetry, the potential at $r = R$ can be written as

   $$\Phi(r = R, \cos\theta) = \sum_\ell a_\ell P_\ell(\cos\theta).$$

   Find C_ℓ in the expression for a_ℓ of the form,

   $$a_\ell = C_\ell \int_{-1}^{1} dx\ \Phi(r = R, x) P_\ell(x).$$

   Here are some identities you might find useful:

   $$\begin{aligned}
   P_0(x) &= 1, \\
   P_1(x) &= x, \\
   P_\ell(x = 1) &= 1, \\
   \sum_\ell (2\ell + 1) P_\ell(x) P_\ell(x') &= 2\delta(x - x'), \\
   \int_{-1}^{1} dx\ P_\ell(x) P_{\ell'}(x) &= \frac{2}{2\ell + 1}\delta_{\ell\ell'}, \\
   \sum_\ell (2\ell + 1) P_\ell(x) P_\ell(x') &= 2\delta(x - x'), \\
   (2\ell + 1) P_\ell(x) &= \frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right], \\
   (\ell + 1) P_{\ell+1}(x) &= (2\ell + 1) x P_\ell(x) - \ell P_{\ell-1}(x), \\
   P_\ell(x) &= \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell \quad \text{(Rodriguez formula)}.
   \end{aligned}$$

   (b) Find a_ℓ for all ℓ for the potential

   $$\Phi(r = R, \cos\theta) = V_0 \cos(2\theta).$$

   Assuming the inside of the sphere is empty, write the potential $\Phi(\vec{r})$ for all $\vec{r}$.

2. Like the previous problem, but with the potential

$$\Phi(r = R, \cos\theta) = \begin{cases} V_0, & \cos\theta > 0 \\ -V_0 & \cos\theta < 0 \end{cases}$$

(a) Using the identities from the previous problem, show that for this potential

$$a_\ell = V_0 P_{\ell-1}(x=0)\frac{(2\ell+1)}{(\ell+1)}.$$

(b) Again, using the identities above, show that

$$\begin{aligned} P_{\ell+1}(x=0) &= -\frac{\ell}{(\ell+1)}P_{\ell-1}(x=0), \\ P_{\ell-1}(x=0) &= -\frac{(\ell-2)}{(\ell-1)}P_{\ell-3}(x=0). \end{aligned}$$

(c) Putting these together, show that

$$\begin{aligned} a_\ell &= -a_{\ell-2}\frac{(2\ell+1)(\ell-2)}{(\ell+1)(2\ell-3)}, \\ a_1 &= 3V_0/2, \quad a_{(\text{even})} = 0. \end{aligned}$$

(d) To test your answer, write a short program to calculate $\Phi(r = R)$ and see whether it matches the expectation.

For solutions, go to course website HW solutions to numbers 11 and 12 in Chapter 4.