Problem Set #1

due Friday, Sep. 1

PHYSICS 851, FALL 2000

1. Consider the matrix:
   \[ \mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]
   (a) What are the eigenvalues of \( \mathcal{M} \)?
   (b) What are the eigenvectors of \( \mathcal{M} \)?

2. Consider the 2\( \times \)2 matrix
   \[ \mathcal{K} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
   (a) What are the eigenvalues of \( \mathcal{K} \)?
   (b) What are the eigenvectors of \( \mathcal{K} \)?

3. A beam of light with wavelength 660 nm is sent along the z axis through a polaroid filter that passes only x polarized light. The beam is initially polarized at 30° to the x axis, and the total energy of the pulse is exactly 10 Joules. Estimate the fluctuations of the energy of the transmitted beam, \( \langle (E - \bar{E})^2 \rangle^{1/2} \). Express the fluctuations as a fraction of the average transmitted energy. (Hint: Consider the binomial distribution, with \( N \) tries with probability \( p \) of success of each try.)

4. Considering light moving along the z axis and using the following definitions for \( |R\rangle \) and \( |L\rangle \) in terms of x and y polarized light,
   \[ |R\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \quad |L\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle), \]
   (a) In terms of \( |R\rangle \) and \( |L\rangle \) write the states \( |45\rangle \) and \( |135\rangle \) which are linearly polarized at 45° and 135° relative to the x axis.
   (b) Calculate the 2 \( \times \) 2 transformation matrix from the 45, 135 basis to the \( RL \) basis.
   (c) Show that this transformation is unitary.
5. The probability that a photon in state $|\Psi\rangle$ passes through an $x$-polaroid is the average value of a physical observable which might be called the \textit{x-polarizedness}.

(a) Write down the operator $P_x$ corresponding to the observable as a matrix in the $xy$ representation. $\langle \Psi | P_x | \Psi \rangle$ is the probability that $|\Psi\rangle$ makes it through the filter.

(b) What are its eigenvalues and eigenstates?

(c) Write the matrix in the $RL$ basis, and show that the eigenvalues are the same as in the $xy$ basis.

6. The trace of a matrix $A$ is defined as:

$$
\text{Tr} A \equiv \sum_i A_{ii}
$$

(a) Show that the trace of $A$ is invariant under a transformation of basis,

$$
A \rightarrow U^\dagger A U
$$

(b) Show that $\text{Tr} AB = \text{Tr} BA$.

7. A plane polarized photon at $\theta = 45^\circ$ enters a special crystal with indices of refraction:

$n_x = 1.50$ for photons polarized along the $x$ axis

$n_y = 1.52$ for photons polarized along the $y$ axis.

Assuming the wavelength of the light is 660 $nm$ before it enters the crystal, choose the thickness of the crystal such that the outgoing light is right circularly polarized. Assume the dispersion is linear, $k = n \omega/c$. 

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