Formulae for Inclusive Electron Scattering

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Abstract
A collection of formulae for kinematics and cross sections etc.

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1 Units
The speed of light $c$ is set equal to 1. To retrieve physical units:

\begin{align*}
\text{energy } E & \rightarrow E \rightarrow E/c \quad (1) \\
\text{mass } m & \rightarrow mc^2 \rightarrow mc \quad (2) \\
\text{4-momentum } p & \rightarrow pc \rightarrow p \quad (3) \\
\text{momentum } \vec{p} & \rightarrow \vec{pc} \rightarrow \vec{p} \quad (4) \\
\text{invariant mass squared } W^2 & \rightarrow W^2c^4 \rightarrow W^2c^2 \quad (5) \\
\text{4-momentum transferred squared } Q^2 & \rightarrow Q^2c^2 \rightarrow Q^2 \quad (6)
\end{align*}

2 Kinematics
We’ll use the notation of MT69 [?]. All noninvariant quantities are evaluated in the lab frame of reference. In this frame, the target particle is initially at rest. The 4-momenta of the incoming and outgoing electron are $s$ and $p$. The 4-momentum of the virtual photon is $q$. The 4-momentum of the target particle is $p_t$ and the total 4-momentum of the products is $p_f$. They are depicted in Fig. (1) and are related to each other by the conservation of energy and momentum:

\begin{align*}
s &= (E_s, \vec{p}_s) \quad (7) \\
p &= (E_p, \vec{p}_p) \quad (8) \\
q &= s - p = (E_s - E_p, \vec{p}_s - \vec{p}_p) = (\nu, \vec{q}) \quad (9) \\
p_t &= (M, 0) \quad (10) \\
p_f &= s + p_t - p = p_t + q = (M + \nu, \vec{q}) \quad (11)
\end{align*}
$s = (E_s, \vec{p}_s)$

$q = (\nu, \vec{q})$

$p = (E_p, \vec{p}_p)$

$p_i = (M, \vec{0})$

$p_f$

Figure 1: Kinematic Variables.
The invariant quantities are the rest masses of the electron \( m \) and target particle \( M \), the imaginary rest mass of the virtual photon \( i \sqrt{Q^2} \), and the sum of the rest masses of the products \( W \):

\[
\begin{align*}
  s^2 &= E_s^2 - |\vec{p}_s|^2 = m^2 \\
  p^2 &= E_p^2 - |\vec{p}_p|^2 = m^2 \\
  q^2 &= \nu^2 - |\vec{q}|^2 = -Q^2 \\
  p_t^2 &= M^2 \\
  p_f^2 &= (M + \nu)^2 - |\vec{q}|^2 = W^2
\end{align*}
\]

The energy and momentum of the incoming and outgoing electron are:

\[
\begin{align*}
  p_s &= |\vec{p}_s|^2 \\
  p_p &= |\vec{p}_p|^2 \\
  E_s &= \sqrt{m^2 + p_s^2} \\
  E_p &= \sqrt{m^2 + p_p^2}
\end{align*}
\]

The energy lost by the incident electron is:

\[
\nu = E_s - E_p
\]

The 4-momentum transferred squared is:

\[
Q^2 = -(s - p)^2 = -s^2 - p^2 + 2sp = -2m^2 + 2(E_sE_p - \vec{p}_s \cdot \vec{p}_p)
\]

It is related to the angle of the scattered electron \( \theta \):

\[
\begin{align*}
\vec{p}_s \cdot \vec{p}_p &= p_s p_p \cos(\theta) \\
Q^2 &= 2 \left[ E_sE_p - p_s p_p \cos(\theta) - m^2 \right]
\end{align*}
\]

The invariant mass of the products is:

\[
W^2 = (M + \nu)^2 - |\vec{q}|^2 = (M^2 + 2\nu M + \nu^2) - \nu^2 - Q^2
\]

\[
= M^2 + 2\nu M - Q^2
\]

It is useful to find a more direct relationship between \( \nu \) and \( W \). Inserting the equation for \( Q \) and isolating the \( p_p \) term gives:

\[
\frac{W^2 - M^2}{2} + (E_sE_p - m^2) - M (E_s - E_p) = p_s p_p \cos(\theta)
\]

Squaring both sides results in a quadratic equation for \( E_p \):

\[
\begin{align*}
0 &= AE_p^2 - BE_p + C \\
A &= M^2 + m^2 + 2ME_s + p_s^2 \sin^2(\theta) \\
B &= [(M + E_s) (W^2 - M^2) + 2p_s^2 (M + E_s \sin^2(\theta))] \\
C &= \left(\frac{(W^2 - M^2)^2}{4} + p_s^2 (W^2 - M^2) + p_s^4 \sin^2(\theta)\right) \\
E_p &= \frac{2C}{B + \sqrt{B^2 - 4AC}}
\end{align*}
\]

Writing this in terms of \( \nu \) gives:

\[
\nu = \frac{E_s (B + \sqrt{B^2 - 4AC}) - 2C}{B + \sqrt{B^2 - 4AC}}
\]
3 Kinematic Limits

The minimum energy of the scattered electron is its mass:

\[ E_{p}^{\min} = m \]  

(34)

Elastic scattering occurs when target particle remains intact. The conservation of momentum requires that the electron loses some energy to the recoil of the target particle. Setting \( W = M \) in equations (30) & (31) gives for \( B \& C \):

\[ B_{el} = 2p_{s}^{2}(M + E_{s}\sin^{2}(\theta)) \]  

(35)

\[ C_{el} = p_{s}^{4}\sin^{2}(\theta) \]  

(36)

Using these elastic values for \( B \& C \) and rearranging some things gives for \( E_{p}^{\max} \):

\[ E_{p}^{\max} = E_{s}\left(1 + \cos(\theta)\sqrt{1 - \frac{m^{2}}{M^{2}}\sin^{2}(\theta)} + [1 + \cos(\theta)]\frac{2m^{2}}{M}\sin^{2}\left(\frac{\theta}{2}\right)\right) \]  

(37)

Consequently, the minimum energy lost by the electron also depends on the scattering angle:

\[ \nu_{\text{min}} = E_{s} - E_{p}^{\max} \]  

(38)

\[ = E_{s}\left(1 + \cos(\theta)\sqrt{1 - \frac{m^{2}}{M^{2}}\sin^{2}(\theta)} + [1 + \cos(\theta)]\frac{2m^{2}}{M}\sin^{2}\left(\frac{\theta}{2}\right)\right) \]  

(39)

The maximum energy lost by the incident electron is:

\[ \nu_{\text{max}} = E_{s} - E_{p}^{\min} = E_{s} - m \]  

(40)

The lowest \( Q^{2} \) occurs when the electron loses most of it’s energy:

\[ Q_{\text{min}}^{2} = 2m\nu_{\text{max}} = 2m(E_{s} - m) \]  

(41)

The highest \( Q^{2} \) occurs when the electron is scattered elastically:

\[ Q_{\text{max}}^{2} = 2M\nu_{\text{min}} \]  

(42)

The lowest \( W \) also occurs for elastic scattering:

\[ W_{\text{min}}^{2} = M^{2} + 2\nu_{\text{min}}M - Q_{\text{max}}^{2} = M^{2} \]  

(43)

The highest \( W \) occurs when the electron loses most of its energy:

\[ W_{\text{max}}^{2} = M^{2} + 2\nu_{\text{max}}M - Q_{\text{min}}^{2} = M^{2} + 2(E_{s} - m)(M - m) \]  

(44)

4 Nonrelativistic Limit

The nonrelativistic limit is reached when \( m \gg p \). The energy and momentum of the incoming and outgoing electron are:

\[ E_{s} \approx m + \frac{p_{s}^{2}}{2m} - \frac{p_{s}^{4}}{8m^{3}} \]  

(45)

\[ E_{p} \approx m + \frac{p_{p}^{2}}{2m} - \frac{p_{p}^{4}}{8m^{3}} \]  

(46)
The energy lost by the incident electron is:

$$\nu \simeq \frac{p_s^2 - p_p^2}{2m} + \frac{p_p^4 - p_s^4}{8m^3}$$  \hfill (47)

The 4-momentum transferred squared is:

$$Q^2 \simeq (\vec{p}_s - \vec{p}_p)^2 - 2 \left( \frac{p_s^2 - p_p^2}{2m} \right)$$  \hfill (48)

The invariant mass of the products is:

$$W^2 \simeq M^2 + \frac{M}{m} \left( p_s^2 - p_p^2 + \frac{p_p^4 - p_s^4}{4m^2} \right) - (\vec{p}_s - \vec{p}_p)^2 + 2 \left( \frac{p_s^2 - p_p^2}{2m} \right)^2$$  \hfill (49)

## 5 Relativistic Limit

The relativistic limit is reached when \( m \ll p \). The energy and momentum of the incoming and outgoing electron are:

$$E_s \simeq \frac{p_s^2}{2m} \simeq p_s$$  \hfill (50)

$$E_p \simeq \frac{p_p^2}{2m} \simeq p_p$$  \hfill (51)

The energy lost by the incident electron is:

$$\nu \simeq (p_s - p_p) \left( 1 - \frac{m^2}{2p_sp_p} \right) \simeq E_s - E_p$$  \hfill (52)

The 4-momentum transferred squared is:

$$Q^2 \simeq 4p_sp_p \left[ \sin^2 \left( \frac{\theta}{2} \right) + \left( \frac{m [p_s - p_p]}{2p_sp_p} \right)^2 \right] \simeq 4E_sE_p \sin^2 \left( \frac{\theta}{2} \right)$$  \hfill (53)

The invariant mass of the products is still:

$$W^2 = M^2 + 2\nu M - Q^2$$  \hfill (54)

The minimum energy of the scattered electron is its mass:

$$E_p^{\text{min}} = m$$  \hfill (55)

In this relativistic limit, the following relation holds for elastic scattering \( W = M \):

$$W^2 = M^2 + 2\nu M - Q^2 \rightarrow Q^2 = 2M\nu$$  \hfill (56)

Rearranging some things gives for \( E_p^{\text{max}} \):

$$E_p^{\text{max}} \simeq \frac{E_s}{1 + \frac{2E_s^2}{M^2} \sin^2 \left( \frac{\theta}{2} \right)}$$  \hfill (57)

Consequently, the minimum energy lost by the electron also depends on the scattering angle:

$$\nu_{\text{min}} = E_s - E_p^{\text{max}} \simeq \frac{2E_s^2 \sin^2 \left( \frac{\theta}{2} \right)}{1 + \frac{2E_s^2}{M^2} \sin^2 \left( \frac{\theta}{2} \right)}$$  \hfill (58)
The maximum energy lost by the incident electron is still:

$$\nu_{\text{max}} = E_s - E_{\text{p}}^{\text{min}} = E_s - m$$  \hspace{1cm} (59)

The lowest $Q^2$ occurs when the electron loses most of it’s energy and is still:

$$Q_{\text{min}}^2 = 2m\nu_{\text{max}} = 2m(E_s - m)$$  \hspace{1cm} (60)

The highest $Q^2$ occurs when the electron is scattered elastically:

$$Q_{\text{max}}^2 = 2M\nu_{\text{min}} \simeq \frac{4E_s^2 \sin^2(\frac{\theta}{2})}{1 + \frac{2E_s}{M} \sin^2(\frac{\theta}{2})}$$  \hspace{1cm} (61)

The lowest $W$ occurs for elastic scattering and is still:

$$W_{\text{min}}^2 = M^2 + 2\nu_{\text{min}}M - Q_{\text{max}}^2 = M^2$$  \hspace{1cm} (62)

The highest $W$ occurs when the electron loses most of its energy:

$$W_{\text{max}}^2 = M^2 + 2\nu_{\text{max}}M - Q_{\text{min}}^2 \simeq M^2 \left( 1 + \frac{2E_s}{M} \right)$$  \hspace{1cm} (63)