Bosons and fermions

One of the fundamental results of relativistic quantum field theory is that all particles can be classified into two groups.

Bosons: particles with *zero or integer spin* (in units of $\hbar$). Examples: photons, all nuclei with even mass numbers. The wavefunction of a system of bosons is symmetric under the exchange of any pair of particles: $\Psi(\ldots,Q_j,\ldots Q_i,..)= \Psi(\ldots,Q_i,\ldots Q_j,..)$. The number of bosons in a given state is *unlimited*.

Fermions: particles with *half-integer spin* (e.g., electrons, quarks, all nuclei with odd mass numbers); the wavefunction of a system of fermions is anti-symmetric under the exchange of any pair of particles: $\Psi(\ldots,Q_j,\ldots Q_i,..)= -\Psi(\ldots,Q_i,\ldots Q_j,..)$. The number of fermions in a given state is *zero or one* (*the Pauli exclusion principle*).

In the early 20th century it became evident that atoms and molecules with even numbers of electrons are more chemically stable than those with odd numbers of electrons. In 1922, Niels Bohr updated his model of the atom by assuming that certain numbers of electrons (for example 2, 8 and 18) corresponded to stable "closed shells".

Pauli looked for an explanation for these numbers, which were at first only empirical. He found an essential clue in a 1924 paper by Edmund C. Stoner which pointed out that for a given value of the principal quantum number (n), the number of energy levels of a single electron in the alkali metal spectra in an external magnetic field, where all degenerate energy levels are separated, is equal to the number of electrons in the closed shell of the noble gases for the same value of n. This led Pauli to realize that the complicated numbers of electrons in closed shells can be reduced to the simple rule of one electron per state, if the electron states are defined using four quantum numbers. For this purpose he introduced a new two-valued quantum number, identified by Goudsmit and Uhlenbeck as electron spin.

Wolfgang Pauli
(1900-1958)

The spin-statistics relation was first formulated in 1939 by Markus Fierz, and was rederived in a more systematic way by Wolfgang Pauli in 1940. A more conceptual argument was provided by Julian Schwinger in 1950.

The Bose or Fermi character of composite objects: the composite objects that have even number of fermions are bosons and those containing an odd number of fermions are themselves fermions.

(an atom of ^{3}He = 2 electrons + 2 protons + 1 neutron ⇒ hence ^{3}He atom is a fermion)

In general, if a neutral atom contains an odd # of neutrons then it is a fermion, and if it contains en even # of neutrons then it is a boson.

The difference between fermions and bosons is specified by the possible values of $\boldsymbol{n_i}$:

fermions: $n_i = 0$ or 1 bosons: $n_i = 0, 1, 2, ...$

distinguish.	particles	Bose	statistics	Fermi	statistics
n_1	n_2	n_1	n_2	n_1	n_2
1	1	1	1		
2	1	2	1	2	1
1	2				
2	2	2	2		
3	1	3	1	3	1
1	3				
3	2	3	2	3	2
2	3				
3	3	3	3		
4	1	4	1	4	1
1	4				
4	2	4	2	4	2
2	4				
4	3	4	3	4	3
3	4				

Consider two non-interacting particles in a 1D box of length ***L***. The total energy is given by

$$E_{n_1,n_2} = \frac{h^2}{8mL^2}\left(n_1^{\ 2} + n_2^{\ 2}\right)$$

The Table shows all possible states for the system with the total energy

$$n_1^{\ 2} + n_2^{\ 2} \le 25$$

Problem (partition function, fermions) (1)

Calculate the partition function of an ideal gas of N=3 identical fermions in equilibrium with a thermal reservoir at temperature T. Assume that each particle can be in one of four possible states with energies ε_1, ε_2, ε_3, and ε_4. (Note that N is fixed).

ε_1	1	1	1	0
ε_2	1	0	1	1
ε_3	1	1	0	1
ε_4	0	1	1	1

a state with E_i

the number of particles in the single-particle state

The Pauli exclusion principle leaves only four accessible states for such system. (The spin degeneracy is neglected).

The partition function (canonical ensemble):

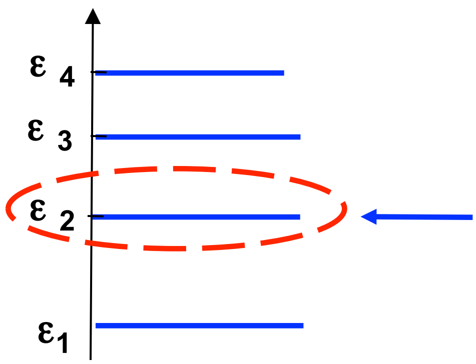
$$Z_3 = \sum_{E_i} \exp\{-\beta E_i\}$$

$$= \exp\{-\beta[\varepsilon_1 + \varepsilon_2 + \varepsilon_3]\} + \exp\{-\beta[\varepsilon_1 + \varepsilon_3 + \varepsilon_4]\}$$

$$+ \exp\{-\beta[\varepsilon_1 + \varepsilon_2 + \varepsilon_4]\} + \exp\{-\beta[\varepsilon_2 + \varepsilon_3 + \varepsilon_4]\}$$

Problem (partition function, fermions) (2)

Calculate the grand partition function of an ideal gas of fermions in equilibrium with a thermal and particle reservoir (T, μ). Fermions can be in one of four possible states with energies ε_1, ε_2, ε_3, and ε_4. (Note that N is not fixed).



each level ε_l is a sub-system independently "filled" by the reservoir

$$\mathcal{Z} = \prod_i \mathcal{Z}_i$$

$$\mathcal{Z}_i = \sum_{n_i=0,1} e^{-\beta n_i(\varepsilon_i - \mu)} = 1 + e^{-\beta(\varepsilon_i - \mu)}$$

$$\mathcal{Z} =$$

$$= \prod_i \left[1 + \exp\left\{\beta\left(\mu - \varepsilon_i\right)\right\}\right]$$

$$= \left[1 + \exp\left\{\beta\left(\mu - \varepsilon_1\right)\right\}\right]\left[1 + \exp\left\{\beta\left(\mu - \varepsilon_2\right)\right\}\right]\left[1 + \exp\left\{\beta\left(\mu - \varepsilon_3\right)\right\}\right]\left[1 + \exp\left\{\beta\left(\mu - \varepsilon_4\right)\right\}\right]$$

$$= 1 + \exp\left\{\beta\left(\mu - \varepsilon_1\right)\right\} + \exp\left\{\beta\left(\mu - \varepsilon_2\right)\right\} + \exp\left\{\beta\left(\mu - \varepsilon_3\right)\right\} + \exp\left\{\beta\left(\mu - \varepsilon_4\right)\right\}$$

$$+ \exp\left\{\beta\left(2\mu - \varepsilon_1 - \varepsilon_2\right)\right\} + \exp\left\{\beta\left(2\mu - \varepsilon_2 - \varepsilon_3\right)\right\} + \ldots$$

Fermi-Dirac distribution

$$\mathcal{Z} = \prod_i \mathcal{Z}_i$$

$$\mathcal{Z}_i = \sum_{n_i=0,1} e^{-\beta n_i(\varepsilon_i - \mu)} = 1 + e^{-\beta(\varepsilon_i - \mu)}$$

The probability of a state to be occupied by a fermion:

$$P(\varepsilon_i, n_i) = \frac{1}{\mathcal{Z}_i} e^{-\beta n_i(\varepsilon_i - \mu)} \quad n_i = 0, 1$$

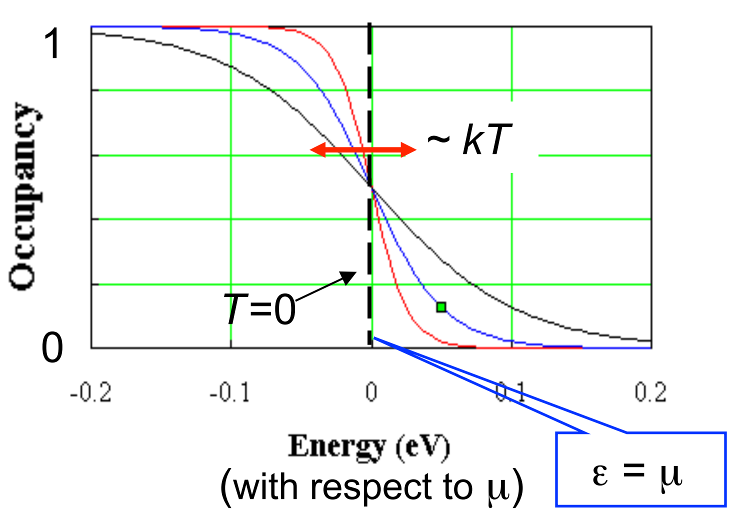
The mean number of fermions in a particular state:

$$\langle n_i \rangle = \frac{1}{\beta \mathcal{Z}_i} \frac{\partial}{\partial \mu} \mathcal{Z}_i = \frac{1}{1 + e^{\beta(\varepsilon_i - \mu)}}$$

$$\langle n \rangle_{\mathrm{FD}}(\varepsilon) = \frac{1}{1 + e^{\frac{\varepsilon - \mu}{kT}}}$$

(μ is determined by T and the particle density)

$$\sum_i \langle n \rangle_{\mathrm{FD}}(\varepsilon_i) = \langle N \rangle$$



At T = 0, all the states with $\varepsilon < \mu$ have the occupancy = 1, all the states with $\varepsilon > \mu$ have the occupancy = 0 (i.e., they are unoccupied). With increasing T, the step-like function is "smeared" over the energy range ~ kT.