In principle, resonances and decaying particles are different entities. Usually, resonance refers to the energy distribution of the outgoing particles in a scattering process, and it is characterized by its energy and width. A decaying state is described in a time dependent setting by its energy and lifetime. Both concepts are related by:

$$T_0 = \frac{\hbar}{\Gamma}$$

This relation has been checked in numerous precision experiments.

See more discussion in
R. de la Madrid,
Nucl. Phys. A812, 13 (2008)

TABLE III. Recent theoretical and experimental lifetimes $\tau$ for NaI $3p\ ^2P_{1/2}$ and $^2P_{3/2}$ and total line strengths $S(3s\text{-}3p)$ (uncertainties given in parentheses).

| Ref. | Method | $J$ | $\tau_J$ (ns) | $S$ (a.u.) |
|---|---|---|---|---|
| | Theoretical | | | |
| [6] | Semiempirical | | | 37.03 |
| [7] | Semiempirical | | | 37.19 |
| [4] | RMBPT all orders | | | 37.38(11) |
| [11] | Coupled clusters | | | 37.56[a] |
| [3] | MCHF-CCP | | | 37.30[a] |
| [5] | MCHF+CI | | | 37.26[a] |
| | Experimental | | | |
| [1] | BGLS | 1/2 | 16.40(3) | 37.04(7)[b] |
| [16] | Pulsed laser | 1/2 | 16.35(6) | 37.15(14)[b] |
| [17] | $C_3$ analysis | 1/2 | 16.31(6) | 37.24(12)[b] |
| [18] | Linewidth | 3/2 | 16.237(35) | 37.30(8)[b] |
| This | BGLS | 1/2 | 16.299(21) | 37.26(5)[c] |
| work | | 3/2 | 16.254(22) | |

[a]Corrected for relativistic effects (−0.09 a.u.) using the ratio between DF and HF values. The original value of Ref. [3] without relativistic correction is 37.39 a.u.
[b]A line strength ratio between the two fine-structure components of 0.5 was assumed in the calculation.
[c]The ratio of the line strengths of the two fine-structure components was determined to 0.50014(44). This is in excellent agreement with the nonrelativistic prediction of 0.5. In the uncertainty estimate for the ratio all those systematical effects were omitted which affect both lifetimes in the same way.

U. Volz et al., Phys. Rev. Lett 76, 2862(1996)

# A comment on the time scale…

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

Time Dependent Schrödinger Equation

$$T_{1/2} = \ln 2 \frac{\hbar}{\Gamma}, \qquad \hbar = 6.58 \cdot 10^{-22}\ \mathrm{MeV} \cdot \mathrm{sec}$$

Can one calculate $\Gamma$ with sufficient accuracy using TDSE?

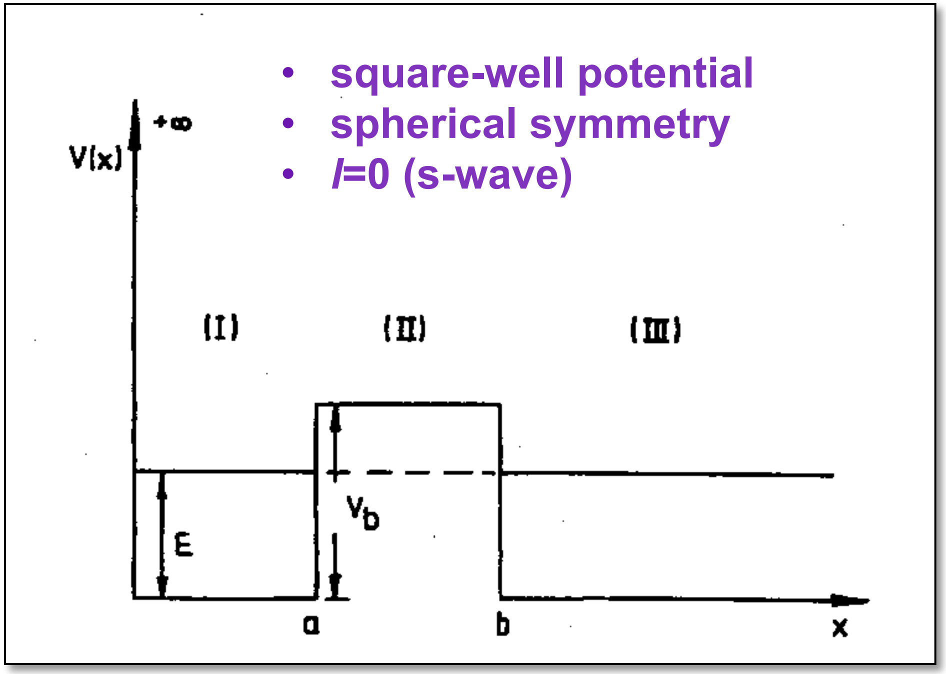
$$T_{s.p.} \approx 3 \cdot 10^{-22}\ \mathrm{sec} = 3\ \mathrm{baby\,sec}$$

$^{238}U$: $T_{1/2}=10^{16}$ years

$^{256}Fm$: $T_{1/2}$=3 hours

For very narrow resonances, explicit time propagation impossible!

# How do resonances appear? A square well example




Radial Schrödinger equation

$$\chi'' + \frac{2M}{\hbar^2}(E - V)\chi = 0 \quad (\chi = \varphi r)$$

Region I:

$$\chi_I = A \sin pr, \quad p^2 = \frac{2ME}{\hbar^2}$$

Region II:

$$\chi_{II} = c_+ e^{q(r-a)} + c_- e^{-q(r-a)}, \quad q^2 = \frac{2M(V_b - E)}{\hbar^2}$$

Region III:

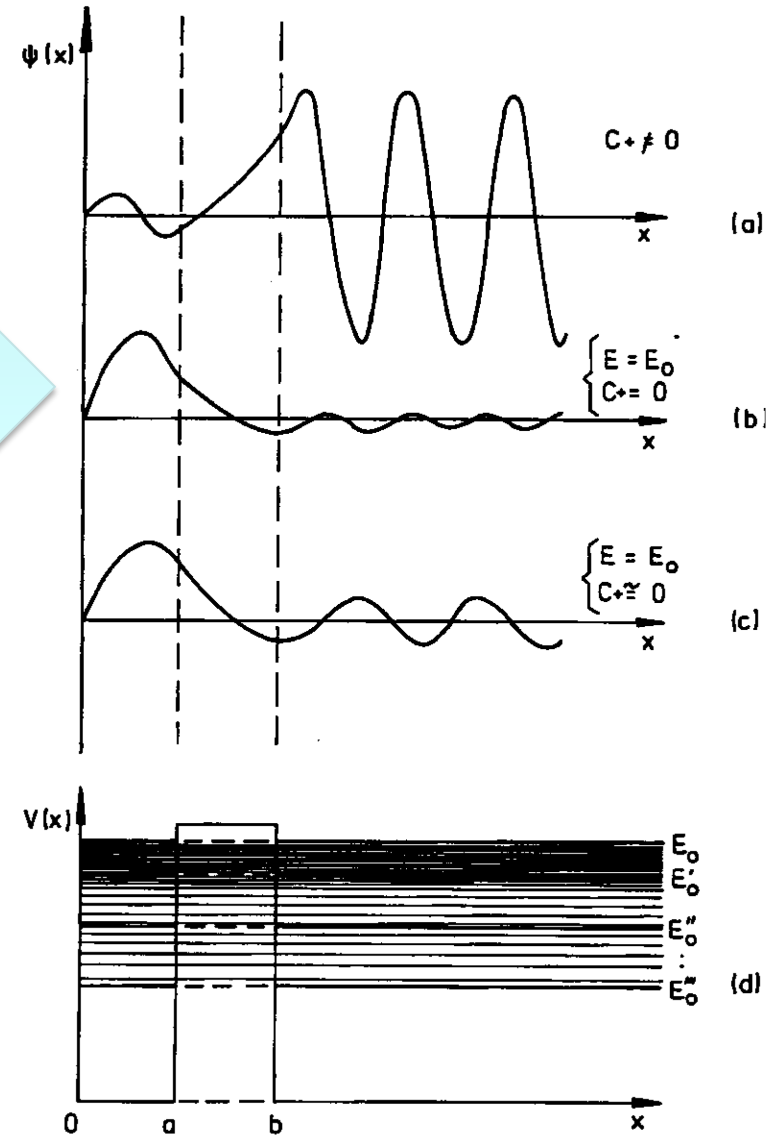
$$\chi_{III} = c_1 e^{ip(r-b)} + c_2 e^{-ip(r-b)}$$

In almost all cases $|\chi_{III}|$ is much larger than $|\chi_I|$. We are now interested in those situations where $|\chi_{III}|$ is as small as possible.

The condition

$$c_+ = 0 \Rightarrow \tan(pa) = -\frac{p}{q}$$

defines "virtual" levels in region I: particle is well localized; very small penetrability through the barrier



When $c_+$=0, the penetrability becomes proportional to

$$\frac{|\chi_{III}|^2}{|\chi_I|^2} \propto \exp\left[-\frac{2}{\hbar}\sqrt{2M(V_b - E)}(b-a)\right]$$

This is the semi-classical WKB result:

$$P = \frac{|\chi_{III}|^2}{|\chi_I|^2} \propto \exp\left[-2\int_{r_1}^{r_2} k(r)\,dr\right]$$

# Width of a narrow resonance

open → $H(t)$, closed → $H_0$

$$H(t) = H_0 + V(t) \quad (|V| << |H_0|)$$ time-dependent Hamiltonian

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$ …expansion of $\psi$ in the basis of $H_o$

$$\hat{H}_0\phi_n = E_n\phi_n \Rightarrow \psi = \sum_n c_n(t)\phi_n e^{-iE_n t/\hbar}$$

$$i\hbar\frac{dc_k}{dt} = \sum_n c_n(t)\langle\phi_k|V|\phi_n\rangle e^{i\omega_{kn}t}, \quad \omega_{kn} = (E_k - E_n)/\hbar$$

As initial conditions, let us assume that at $t=0$ the system is in the state $\phi_0$

$$c_n(0) = \begin{cases} 1 \text{ for n} = 0 \\ 0 \text{ for n} \neq 0 \end{cases}$$

If the perturbation is weak, in the first order, we obtain:

$$i\hbar\frac{dc_k}{dt} = \langle\phi_k|V|\phi_0\rangle e^{i\omega_{k0}t}$$

Furthermore, if the time variation of $V$ is slow compared with exp$(i\omega_{ko}t)$, we may treat the matrix element of V as a constant. In this approximation:

$$c_k(t) = \frac{\langle \phi_k | V | \phi_0 \rangle}{E_k - E_0}\left(1 - e^{i\omega_{k0}t}\right)$$

The probability for finding the system in state $k$ at time t if it started from state 0 at time t=0 is:

$$|c_k(t)|^2 = 2\frac{\left|\langle \phi_k | V | \phi_0 \rangle\right|^2}{\left(E_k - E_0\right)^2}\left(1 - \cos\omega_{k0}t\right)$$

The total probability to decay to a group of states within some interval labeled by $f$ equals:

$$\sum_{k\in f} |c_k(t)|^2 = \frac{2}{\hbar^2}\int \frac{\left|\langle \phi_k | V | \phi_0 \rangle\right|^2}{\omega_{k0}^2}\left(1 - \cos\omega_{k0}t\right)\rho\left(E_k\right)dE_k$$

The transition probability per unit time is

$$\mathcal{W} = \frac{d}{dt}\sum_{k\in f}\left|c_k(t)\right|^2 = \frac{2}{\hbar^2}\int\left|\left\langle\phi_k\right|V\left|\phi_0\right\rangle\right|^2 \frac{\sin\omega_{k0}t}{\omega_{k0}}\rho\left(E_k\right)dE_k$$

Since the function sin*(x)/x* oscillates very quickly except for *x*~0, only small region around $E_0$ can contribute to this integral. In this small energy region we may regard the matrix element and the state density to be constant. This finally gives:

$$\mathcal{W}_{0\to f} = \frac{2\pi}{\hbar}\left|\left\langle\phi_f\right|V\left|\phi_0\right\rangle\right|^2\rho\left(E_f\right)$$

Fermi's golden rule

Although named after Fermi, most of the work leading to the Golden Rule was done by Dirac, who formulated an almost identical equation 1927. It is given its name because Fermi called it "Golden Rule No. 2." in 1950.

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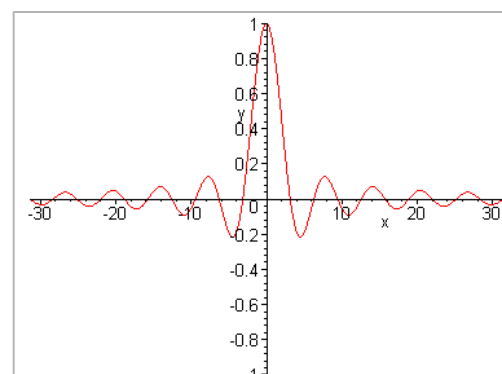
$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$$



$$E = E_0 - i\frac{\Gamma}{2}; \quad \Gamma = \hbar w \qquad N = N_0 e^{-wt}$$

mean lifetime $$T_0 = \frac{\hbar}{\Gamma}$$

half-life $$T_{1/2} = \ln 2\,\frac{\hbar}{\Gamma} = \ln 2\, T_0 \qquad \frac{N_0}{2} = N_0 e^{-wT_{1/2}}$$

transition probability $$\mathcal{W}_{0\to f} = \frac{1}{T_{0\to f}} = \frac{\Gamma_{0\to f}}{\hbar}$$

Fermi's golden rule $$\Gamma_{0\to f} = 2\pi \left|\langle \phi_f | V | \phi_0 \rangle\right|^2 \rho(E_f)$$

$$\psi(t) = \frac{\psi(0)}{2\pi}\int c(E) e^{-iEt/\hbar} dE$$

$$c(E) = \int_0^\infty e^{i(E-E_0+i\Gamma/2)t/\hbar} dt = \frac{i\hbar}{E - E_0 + i\Gamma/2}$$

normalized amplitude $$|c(E)|^2 = \frac{1}{\pi}\frac{\Gamma/2}{(E-E_0)^2 + (\Gamma/2)^2}$$

uncertainty principle $$\Gamma T_0 = \hbar$$