A comment on the time scale…

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

Time Dependent Schrödinger Equation

$$T_{1/2} = \ln 2 \frac{\hbar}{\Gamma}, \qquad \hbar = 6.58 \cdot 10^{-22}\ \mathrm{MeV} \cdot \mathrm{sec}$$

Can one calculate Γ with sufficient accuracy using TDSE?

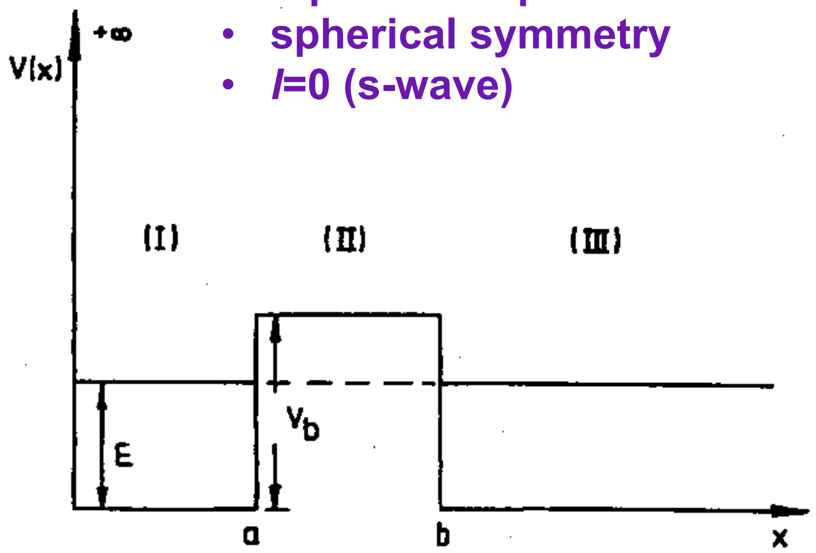
$$T_{s.p.} \approx 3 \cdot 10^{-22}\ \mathrm{sec} = 3\ \mathrm{baby\,sec}$$

^{238}U: $T_{1/2}$=10^{16} years

^{256}Fm: $T_{1/2}$=3 hours

For very narrow resonances, explicit time propagation impossible!

How do resonances appear? A square well example



- square-well potential
- spherical symmetry
- *l*=0 (s-wave)

Radial Schrödinger equation

$$\chi'' + \frac{2M}{\hbar^2}(E - V)\chi = 0 \quad (\chi = \varphi r)$$

Region I:

$$\chi_I = A \sin pr, \quad p^2 = \frac{2ME}{\hbar^2}$$

Region II:

$$\chi_{II} = c_+ e^{q(r-a)} + c_- e^{-q(r-a)}, \quad q^2 = \frac{2M(V_b - E)}{\hbar^2}$$

Region III:

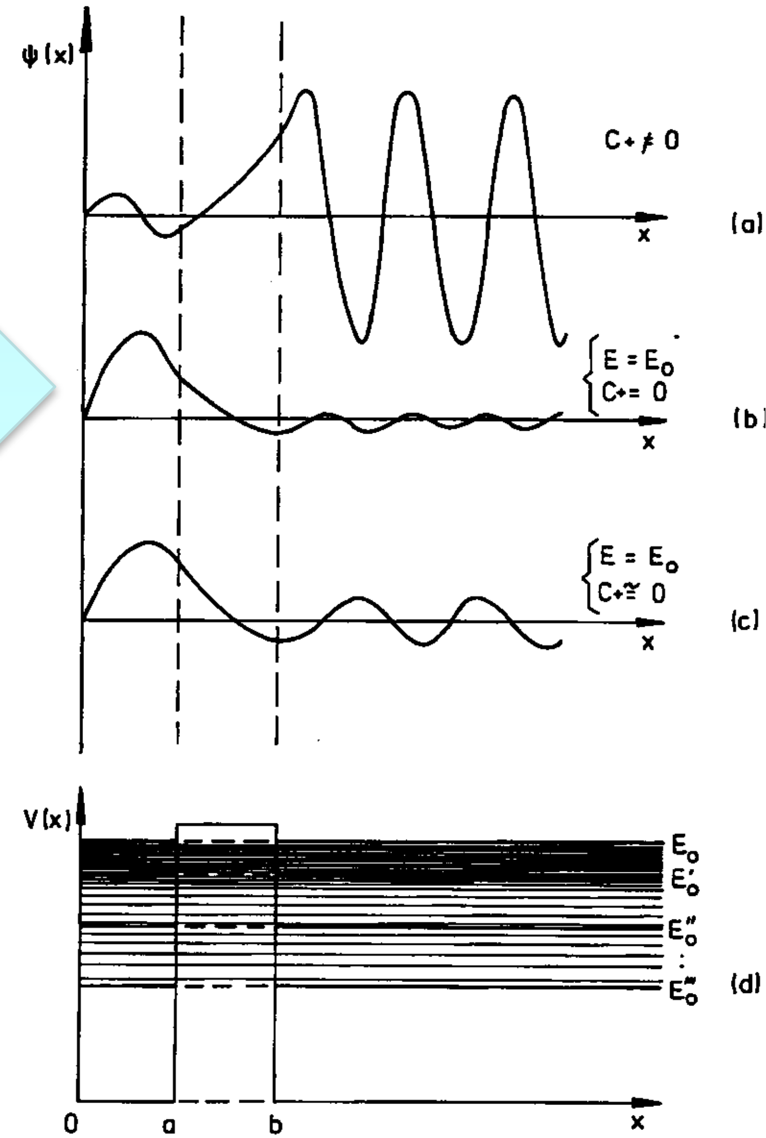
$$\chi_{III} = c_1 e^{ip(r-b)} + c_2 e^{-ip(r-b)}$$

In almost all cases $|\chi_{III}|$ is much larger than $|\chi_I|$. We are now interested in those situations where $|\chi_{III}|$ is as small as possible.

The condition

$$c_+ = 0 \Rightarrow \tan(pa) = -\frac{p}{q}$$

defines "virtual" levels in region I: particle is well localized; very small penetrability through the barrier



When c_+=0, the penetrability becomes proportional to

$$\frac{\left|\chi_{III}\right|^2}{\left|\chi_I\right|^2} \propto \exp\left[-\frac{2}{\hbar}\sqrt{2M(V_b - E)}(b-a)\right]$$

This is the semi-classical WKB result:

$$P = \frac{\left|\chi_{III}\right|^2}{\left|\chi_I\right|^2} \propto \exp\left[-2\int_{r_1}^{r_2} k(r)\,dr\right]$$

Width of a narrow resonance

open closed

$$H(t) = H_0 + V(t) \quad (|V| << |H_0|)$$ time-dependent Hamiltonian

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$ …expansion of ψ in the basis of H_o

$$\hat{H}_0 \phi_n = E_n \phi_n \Rightarrow \psi = \sum_n c_n(t) \phi_n e^{-iE_n t/\hbar}$$

$$i\hbar \frac{dc_k}{dt} = \sum_n c_n(t) \langle \phi_k | V | \phi_n \rangle e^{i\omega_{kn} t}, \quad \omega_{kn} = \left(E_k - E_n\right)/\hbar$$

As initial conditions, let us assume that at $t=0$ the system is in the state ϕ_0

$$c_n(0) = \begin{cases} 1 \text{ for n} = 0 \\ 0 \text{ for n} \neq 0 \end{cases}$$

If the perturbation is weak, in the first order, we obtain:

$$i\hbar \frac{dc_k}{dt} = \langle \phi_k | V | \phi_0 \rangle e^{i\omega_{k0} t}$$

Furthermore, if the time variation of *V* is slow compared with exp$(i\omega_{ko}t)$, we may treat the matrix element of V as a constant. In this approximation:

$$c_k(t) = \frac{\langle \phi_k | V | \phi_0 \rangle}{E_k - E_0}\left(1 - e^{i\omega_{k0}t}\right)$$

The probability for finding the system in state *k* at time t if it started from state 0 at time t=0 is:

$$|c_k(t)|^2 = 2\frac{|\langle \phi_k | V | \phi_0 \rangle|^2}{(E_k - E_0)^2}\left(1 - \cos\omega_{k0}t\right)$$

The total probability to decay to a group of states within some interval labeled by *f* equals:

$$\sum_{k \in f} |c_k(t)|^2 = \frac{2}{\hbar^2}\int \frac{|\langle \phi_k | V | \phi_0 \rangle|^2}{\omega_{k0}^2}\left(1 - \cos\omega_{k0}t\right)\rho\left(E_k\right)dE_k$$

The transition probability per unit time is

$$\mathcal{W} = \frac{d}{dt}\sum_{k\in f}\left|c_k(t)\right|^2 = \frac{2}{\hbar^2}\int\left|\left\langle\phi_k\middle|V\middle|\phi_0\right\rangle\right|^2\frac{\sin\omega_{k0}t}{\omega_{k0}}\rho\left(E_k\right)dE_k$$

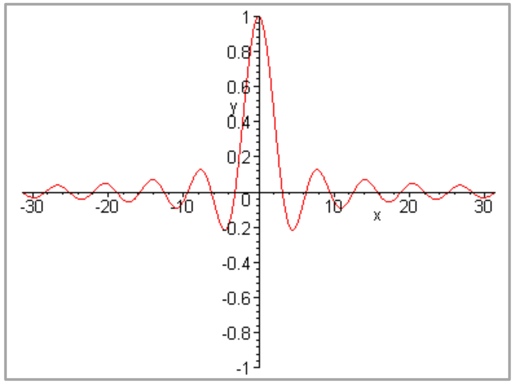
Since the function sin*(x)/x* oscillates very quickly except for *x*~0, only small region around E_0 can contribute to this integral. In this small energy region we may regard the matrix element and the state density to be constant. This finally gives:

$$\mathcal{W}_{0\to f} = \frac{2\pi}{\hbar}\left|\left\langle\phi_f\middle|V\middle|\phi_0\right\rangle\right|^2\rho\left(E_f\right)$$

Fermi's golden rule

Although named after Fermi, most of the work leading to the Golden Rule was done by Dirac, who formulated an almost identical equation 1927. It is given its name because Fermi called it "Golden Rule No. 2." in 1950.

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$$



$$E = E_0 - i\frac{\Gamma}{2}; \quad \Gamma = \hbar w \qquad N = N_0 e^{-wt}$$

mean lifetime $\quad T_0 = \frac{\hbar}{\Gamma}$

half-life $\quad T_{1/2} = \ln 2 \, \frac{\hbar}{\Gamma} = \ln 2 \, T_0 \qquad \frac{N_0}{2} = N_0 e^{-wT_{1/2}}$

transition probability $\quad \mathcal{W}_{0\to f} = \frac{1}{T_{0\to f}} = \frac{\Gamma_{0\to f}}{\hbar}$

Fermi's golden rule $\quad \Gamma_{0\to f} = 2\pi \left| \langle \phi_f | V | \phi_0 \rangle \right|^2 \rho\left(E_f\right)$

$$\psi(t) = \frac{\psi(0)}{2\pi} \int c(E) e^{-iEt/\hbar} dE$$

$$c(E) = \int_0^\infty e^{i(E-E_0+i\Gamma/2)t/\hbar} dt = \frac{i\hbar}{E - E_0 + i\Gamma/2}$$

normalized amplitude $\quad |c(E)|^2 = \frac{1}{\pi} \frac{\Gamma/2}{(E-E_0)^2 + (\Gamma/2)^2}$

uncertainty principle $\quad \Gamma T_0 = \hbar$