Symmetries in quantum mechanics

(see "Symmetry in Physics", J.P. Elliott and P.G. Dawber, The Macmillan Press, London)

- Wave equation for the Hamiltonian operator:

$$\hat{H}\Psi_k = E_k\Psi_k$$

- Group of transformations $\mathcal{G}$ whose elements G commute with H:

$$G\hat{H}G^{-1} = \hat{H}$$

- We say that H is $i$nvariant under $\mathcal{G}$ or totally symmetric with respect to the elements of $\mathcal{G}$
- What are the properties of $G\Psi_k$?
- If we know the properties of $\mathcal{G}$, we can classify the wave functions
- Further, the same group-theoretical structure will tell us about the spectroscopy of the system

Classification of symmetry groups

Point groups: geometric symmetries that keep at least one point fixed

- D_1: (dihedral) reflection group (2 element group: identity and single reflection)
- C_n: cyclic n-fold rotation (C_1 is a trivial group containing identity operation)

Lie groups: continuous transformation groups. Transformations generated by physical operators. The set of commutators between generators is closed. Its *Casimir* operator commuters with all the generators.

- SO(3): group of rotations in 3D (isomorphic with SU(2))
- Poincare group (Translations, Lorentz transformations)

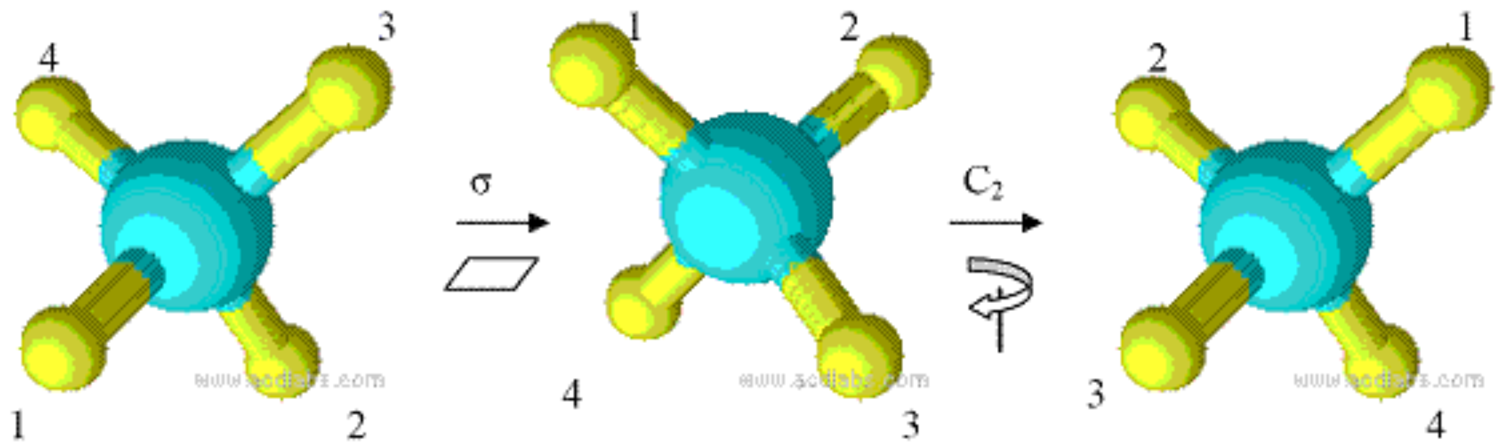
Vector spaces: Scalars, Vectors, Tensors…

Orthogonal transformations U

- preserve lengths of vectors and angles between them
- map orthonormal bases to orthonormal bases
- Orthogonal transformations in two- or three-dimensional Euclidean space are stiff rotations, reflections, or combinations of a rotation and a reflection (also known as improper rotations).

$\det(U)=1$ – usual (stiff) rotations (scalars, vectors,…)

$\det(U)=-1$ – improper rotations (pseudo-scalars, axial vectors, …)



Improper rotation operation S_4 in CH_4

Translational Invariance

$$\vec{r}_k\,' = \vec{r}_k - \vec{a} = U\,\vec{r}_k U^{-1}$$

$$\vec{p}_k\,' = \vec{p}_k, \quad \vec{s}_k\,' = \vec{s}_k$$

$$U(\vec{a}) = \exp\left\{-\frac{i}{\hbar}\vec{a}\vec{P}\right\}$$

Unitary transformations:
$U^{+}=U^{-1}$

Under unitary transformation *U*, an operator *A* transforms as $A'=UAU^{-1}$

- Total momentum (nucleons, mesons, photons, leptons, etc.)
- Transformation *generator*

$$e^{X}Ye^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots$$

For $[X,Y]$ central, i.e., commuting with both *X* and *Y*: $e^{sX}Ye^{-sX} = Y + s[X,Y]$

Time displacement

$$U(t_0) = \exp\left\{\frac{i}{\hbar}t_0\hat{H}\right\}$$

Rotations in 3D (space isotropy)

$$R(\vec{\chi}) = \exp\{-i\vec{\chi}\vec{J}\}$$

$\vec{\chi}$ a set of three angles (a vector) representing rotations along x,y,z

$$[J_x, J_y] = iJ_z \qquad \text{(+ cycl.)}$$

- Total angular momentum
- Transformation generator
- SO(3) or SU(2) group!

Rotational states of the system labeled by the total angular momentum quantum numbers *JM*

see examples of spectra at http://www.nndc.bnl.gov