PHYSICS 231
Oscillations
Hooke’s law

\[ F_s = -kx \quad \text{Hooke’s law} \]

If there is no friction, the mass continues to oscillate back and forth.

If a force is proportional to the displacement \( x \), but opposite in direction, the resulting motion of the object is called: \textit{simple harmonic oscillation}
The spring constant $k$

When the object hanging from the spring is not moving:

\[ F_{\text{spring}} = -F_{\text{gravity}} \]
\[ -kd = -mg \]
\[ k = \frac{mg}{d} \]

$k$ is a constant, so if we hang twice the amount of mass from the spring, $d$ becomes twice larger:

\[ k = \frac{(2m)g}{(2d)} = \frac{mg}{d} \]
Simple harmonic motion

Amplitude (A): maximum distance from equilibrium (unit: m)

Period (T): Time to complete one full oscillation (unit: s)

Frequency (f): Number of completed oscillations per second (unit: 1/s = 1 Herz [Hz])

\[
x(t) = A \cos(\omega t) = A \cos(2\pi f t) = A \cos(2\pi t / T)
\]

\[\omega = 2\pi f = 2\pi / T\]

\[\omega: \text{“angular frequency”} \quad f = 1 / T\]
energy and velocity

\[ E_{\text{kin}} \left( \frac{1}{2}mv^2 \right) \quad E_{\text{pot, spring}} \left( \frac{1}{2}kx^2 \right) \quad \text{Sum} \]

\[ \begin{array}{ccc}
0 & \frac{1}{2}kA^2 & \frac{1}{2}kA^2 \\
\frac{1}{2}mv^2 & 0 & \frac{1}{2}mv^2 \\
0 & \frac{1}{2}k(-A)^2 & \frac{1}{2}kA^2 \\
\end{array} \]

conservation of ME: \( \frac{1}{2}m[v(x=0)]^2 = \frac{1}{2}kA^2 \) so \( v(x=0) = \pm A \sqrt{k/m} \)
velocity more general

Total ME at any displacement \( x \):
\[
\frac{1}{2}mv^2 + \frac{1}{2}kx^2
\]

Total ME at max. displacement \( A \):
\[
\frac{1}{2}kA^2
\]

Conservation of ME:
\[
\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2
\]

So:
\[
v = \pm \sqrt{(A^2 - x^2)k/m}
\]

<table>
<thead>
<tr>
<th>position ( X )</th>
<th>velocity ( V )</th>
<th>acceleration ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+A)</td>
<td>0</td>
<td>(-kA/m)</td>
</tr>
<tr>
<td>0</td>
<td>(\pm A\sqrt{(k/m)})</td>
<td>0</td>
</tr>
<tr>
<td>(-A)</td>
<td>0</td>
<td>(kA/m)</td>
</tr>
</tbody>
</table>
If oscillation is in vertical direction:  also add gravitational PE

\[ ME = KE + PE_{spring} + PE_{gravity} \]

\[ = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mgh \]
example

A mass of 1 kg is hung from a spring. The spring stretches by 0.5 m. Next, the spring is placed horizontally and fixed on one side to the wall. The same mass is attached and the spring stretched by 0.2 m and then released. What is the acceleration upon release?

1st step: find the spring constant k

\[ F_{\text{spring}} = -F_{\text{gravity}} \text{ or } -kd = -mg \]

\[ k = \frac{mg}{d} = \frac{1 \times 9.8}{0.5} = 19.6 \text{ N/m} \]

2nd step: find the acceleration upon release

Newton’s second law: \( F = ma \) \( \Rightarrow -kx = ma \) \( \Rightarrow a = -\frac{kx}{m} \)

\[ a = -19.6 \times 0.2 / 1 = -3.92 \text{ m/s}^2 \]
The harmonic motion can be described with \( x(t) = A \cos(\omega t) \)
if \( T=1 \)  \( f=1 \) then \( \omega = 2\pi \) since \( x(0)=x(1) \) if \( x(t)=A \cos(2\pi t) \)
if \( T=0.5 \)  \( f=2 \) then \( \omega = 4\pi \) since \( x(0)=x(0.5) \) if \( x(t)=A \cos(4\pi t) \)
if \( T=2 \)  \( f=0.5 \) then \( \omega = \pi \) since \( x(0)=x(2) \) if \( x(t)=A \cos(\pi t) \)
\( x(t)=A \cos(\omega t)=A \cos(2\pi ft)=A \cos(2\pi t/T) \quad \omega = 2\pi f = 2\pi /T \)
\( \omega \): “angular frequency”
Extra credit quiz

1) what is the amplitude of the harmonic oscillation?
   a) 2cm  b) 4cm  c) 5cm  d) 10cm

2) what is the period of the harmonic oscillation?
   a) 1s  b) 2s  c) 4s  d) 5s

3) what is the frequency of the harmonic oscillation?
   a) 1 Hz  b) 0.5 Hz  c) 0.25 Hz  d) 0.2 Hz

a) Amplitude: 5cm (0.05 m)
b) period: time to complete one full oscillation: 4s
c) frequency: number of oscillations per second = 1/T = 0.25 Hz
The projection of the position of the circulating object on the x-axis as a function of time is the same as the position of the oscillating spring.

The projection of the linear velocity of the rotating object on the x-axis is the velocity of the mass on the spring.
If \( x(t) \) is maximal, \( v(t) = 0 \)
If \( x(t) = 0 \), the speed is maximal
The curve of \( x(t) \) lags that of \( v(t) \) by \( \frac{1}{4} \) of a period
If \( x(t) \sim \cos(\omega t) \) then \( v(t) \sim -\sin(\omega t) \)
velocity in harmonic motion

1) If \( x(t) \sim \cos(\omega t) \) then \( v(t) \sim -\sin(\omega t) \)

2) We had already seen that...

\[
\begin{array}{|c|c|}
\hline
\text{position } X & \text{velocity } V \\
\hline
+A & 0 \\
0 & \pm A\sqrt{(k/m)} \\
-A & 0 \\
\hline
\end{array}
\]

3) From which we conclude that if \( x=A\cos(\omega t) \) then
\[
v=-A\sqrt{(k/m)}\sin(\omega t)
\]

4) One can derive for a spring that \( \omega^2=k/m \)

5) so that: \( v=-A\omega\sin(\omega t) \)
simple harmonic motion

\[ x_{\text{harmonic}}(t) = A \cos(\omega t) \]
\[ v_{\text{harmonic}}(t) = -\omega A \sin(\omega t) \]

where \( A \) is the amplitude of the oscillation, and \( \omega = \frac{2\pi}{T} = 2\pi f \), where \( T \) is the period of the harmonic motion and \( f = \frac{1}{T} \) the frequency.

for a spring: \( \omega = \sqrt{\frac{k}{m}} \)
Newton’s second law: $F = ma \Rightarrow -kx = ma \Rightarrow a = -\frac{kx}{m} = -\omega^2 x$

displacement vs acceleration

**displacement x**

$A$  

$-A$

**acceleration (a)**

$kA/m$

$-kA/m$

$a(t) = -\omega^2 x(t)$ so $a(t) = -A\omega^2 \cos(\omega t)$

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\[ x_{\text{harmonic}}(t) = A \cos(\omega t) \]

\[ \omega = 2\pi f = \frac{2\pi}{T} = \sqrt{k/m} \]

\[ v_{\text{harmonic}}(t) = -\omega A \sin(\omega t) \]

for the spring \( \omega = \sqrt{k/m} \)

\[ a_{\text{harmonic}}(t) = -\omega^2 A \cos(\omega t) \]
Example

A mass of 0.2 kg is attached to a spring with k=100 N/m. The spring is stretched over 0.1 m and released.

a) What is the angular frequency ($\omega$) of the corresponding circular motion?
b) What is the period (T) of the harmonic motion?
c) What is the frequency (f)?
d) What are the functions for $x$, $v$ and $t$ of the mass as a function of time? Make a sketch of these.

a) $\omega = \sqrt{(k/m)} = \sqrt{(100/0.2)} = 22.4$ rad/s
b) $\omega = 2\pi/T \quad T = 2\pi/\omega = 0.28$ s
c) $\omega = 2\pi f \quad f = \omega/2\pi = 3.55$ Hz ($=1/T$)
d) $x_{\text{harmonic}}(t) = A\cos(\omega t) = 0.1\cos(22.4t)$
   $v_{\text{harmonic}}(t) = -\omega A\sin(\omega t) = -2.24\sin(22.4t)$
   $a_{\text{harmonic}}(t) = -\omega^2 A\cos(\omega t) = -50.2\cos(22.4t)$
Extra credit quiz

A mass is oscillating horizontally while attached to a spring with spring constant $k$. Which of the following is true?

a) When the magnitude of the displacement is largest, the magnitude of the acceleration is also largest.

b) When the displacement is positive, the acceleration is also positive.

c) When the displacement is zero, the acceleration is non-zero.
Another simple harmonic oscillation: the pendulum

Restoring force: \( F = -mg \sin \theta \)

The force pushes the mass \( m \) back to the central position.

\( \sin \theta \approx \theta \) if \( \theta \) is small (<15°) radians!!!

\( F = -mg \theta \) also \( \theta = s/L \)

so: \( F = -(mg/L)s \)
pendulum vs spring

<table>
<thead>
<tr>
<th>parameter</th>
<th>spring</th>
<th>pendulum</th>
</tr>
</thead>
<tbody>
<tr>
<td>restoring force $F$</td>
<td>$F = -kx$</td>
<td>$F = -(mg/L)s$</td>
</tr>
<tr>
<td>period $T$</td>
<td>$T = 2\pi \sqrt{(m/k)}$</td>
<td>$T = 2\pi \sqrt{(L/g)}$ *</td>
</tr>
<tr>
<td>frequency $f$</td>
<td>$f = \sqrt{(k/m)/(2\pi)}$</td>
<td>$f = \sqrt{(g/L)/(2\pi)}$</td>
</tr>
<tr>
<td>angular frequency</td>
<td>$\omega = \sqrt{(k/m)}$</td>
<td>$\omega = \sqrt{(g/L)}$</td>
</tr>
</tbody>
</table>

* $T = 2\pi \sqrt{\frac{m}{mg/L}} = 2\pi \sqrt{\frac{L}{g}}$
example: a pendulum clock

The machinery in a pendulum clock is kept in motion by the swinging pendulum. Does the clock run faster, at the same speed, or slower if:

a) The mass is hung higher
b) The mass is replaced by a heavier mass
c) The clock is brought to the moon
d) The clock is put in an upward accelerating elevator?

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

<table>
<thead>
<tr>
<th></th>
<th>L↓</th>
<th>m↑</th>
<th>moon</th>
<th>elevator</th>
</tr>
</thead>
<tbody>
<tr>
<td>faster</td>
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<tr>
<td>same</td>
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<td>slower</td>
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pendulum

- A pendulum with length $L$ of 1 meter and a mass of 1 kg, is oscillating harmonically. The period of oscillation is $t$ s. If $L$ is increased to 4 m, and the mass replaced by one of 0.5 kg, the period becomes:

- A) $0.5t$ s
- B) $1t$ s
- C) $2t$ s
- D) $4t$ s
- E) $8t$ s

$L \times 4$, then $T \times \sqrt{4} = 2T$

Mass does not matter
A block with mass of 200 g is placed over an opening. A spring is placed under the opening and compressed by a distance \(d=5\) cm. Its spring constant is 250 N/m. The spring is released and launches the block. How high will it go relative to its rest position (\(h\))? (the spring can be assumed massless)

\[
\text{ME} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mgh \quad \text{must be conserved.}
\]

Initial: \(v=0\), \(x=-0.05\) m, \(h=0\)
\[
0.5 \times 0.2 \times 0^2 + 0.5 \times 250 \times 0.05^2 + 0.2 \times 9.8 \times 0 = 0.3125
\]

Final: \(v=0\) (at highest point mass does not move), \(x=0\) (spring no longer compressed), \(h=?\)
\[
0.5 \times 0.2 \times 0^2 + 0.5 \times 250 \times 0.0^2 + 0.2 \times 9.8 \times h = 1.96 \times h
\]

Conservation of ME: \(0.3125 = 1.96 \times h\) \(h=0.16\) m